INTRODUCTION TO
FLUID MECHANICS

By A. DKHAR

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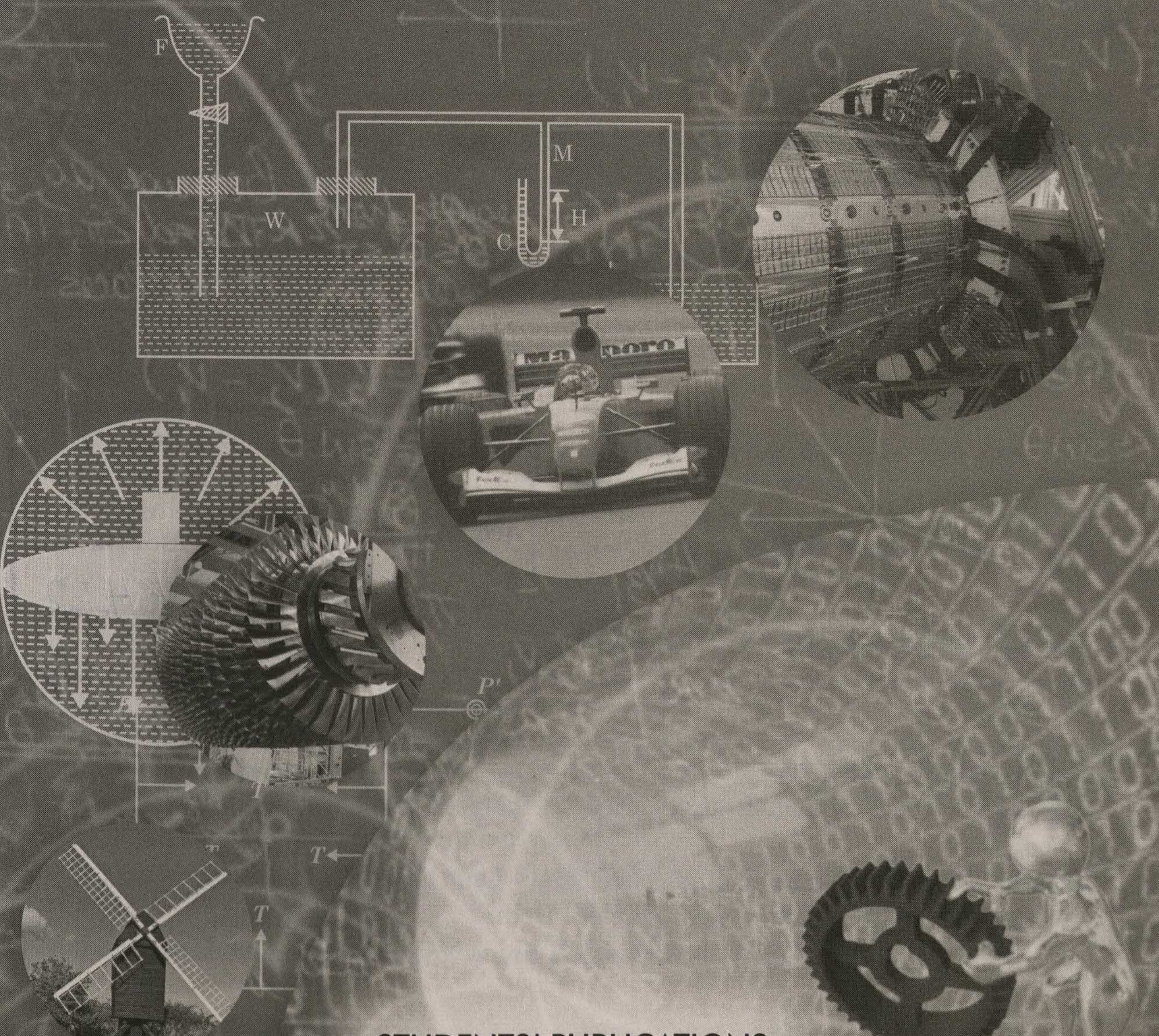

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Preface

The present book has been written taking into account the Syllabus of the North Eastern Hill University for the first year students, and subsequently with the introduction of the semester system, for the students of the first semester. Indeed, there is quite a number of books that which are already available on the topics covered, but it is a humble beginning on my part to try and present the subject matter in a simple, concise manner and taking into account the fact that, the average student needs to have a thorough understanding of the basic principles and concepts before he/she can really have an interest for the subject matter. Although the topics have been covered in most of the syllabi of different boards of Higher Secondary Education, however, particular care have been taken to make the treatment clear and comprehensible to the slightest detail, keeping in mind the fact that the average student may not have been able to grasp the subject matter completely and thoroughly by simply learning about it in the earlier class because of time constrain and many other factors. The mathematical treatment involved in the different sections is, therefore, given in detail and in a way to make the reading easy to understand and follow. In fact, any sincere student will find that the book is a ready material for the University exams and he/she can readily answer any questions set from the various topics presented in the book. The book has been written in such a format, that answers to various questions that may be posed are a ready reference in the sections covered.

Also, keeping in mind, the fact that, a mere knowledge of theory will not fully help the students in solving numerical problems, solved examples are given at the end of the chapters for the students to understand how to go about solving the various types of problems that may be set from the different sections. A number of unsolved numerical and sample questions are also given so that the students can practice and also refer to, respectively.

Although a humble attempt has been made to present the subject matter in a simple and concise manner by keeping in mind the average students, however, no originality is claimed, or indeed can be claimed in a work which is of a specific course of this nature, as the theories and approach to the topics cannot be presented in a completely different format, different from the others. In fact, there is simply no other way.

My thanks are also due to the publishers of this volume who have taken interest and for the meticulous care taken for the publication of this volume in this wonderful form. I also thank the Principal, Shillong College, who has greatly encouraged and expressed his confidence in me to bring out this volume. I am also particularly thankful to Dr. S. Khongwir who has greatly helped me and without whose help this volume might not have been possible. Also, I thank my friends, my colleagues in the department for their relentless support and encouragement. Last but not the least I thank my students, whose sincerity has motivated me in writing this book.

I dedicate this book to my parents and I thank God Almighty for his guidance throughout and ever.

Any suggestions toward the improvement of the book will be gratefully accepted.

A. Dkhar
Author

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Chapter 1

Hydrostatics — Surface Tension

1.1. HYDROSTATICS

Hydrostatics is a branch of Physics which deals with the mechanics of **fluids** in equilibrium. So it is of central importance for us, before we proceed further, to understand exactly, what is a 'fluid'?

Unlike the case of solids in which the strain set up under the application of a shearing (tangential) stress lasts so long as the applied stress is not removed, a fluid is a substance which cannot permanently or indefinitely withstand or oppose a shearing stress. In fact, a fluid constantly and continuously yield to the applied stress, although, the yield may be rapid in some cases and slow in some other cases. In the former case, the fluid is said to be *mobile* (e.g. water, alcohol etc.) and in the latter case, *viscous* (e.g. honey, glycerin etc.). Fluids are broadly divided into two classes, viz. *Liquids* & *Gases.*

A liquid is a fluid which occupies a definite volume which cannot be altered, however great the force applied to compress it, and has no shape of its own. It generally takes up the volume of the container or vessel in which it is poured. When a liquid is poured from one vessel to another of a different shape, its volume does not change but it takes up the shape of the different vessels in which it is poured. In other words, a liquid is a fluid which is quite incompressible and has a free surface of its own. Examples of liquid are water, alcohol, oils, honey, glycerin etc. (Strictly speaking, however, all liquids do get compressed a little when subjected to very high deforming forces of the order of few hundred atmospheres. e.g. when water is subjected to a pressure of about 200 atmosphere, it undergoes a reduction of only one by hundredth part of its original volume).

A gas, on the other hand, is a fluid which has neither a shape nor a free surface of its own. It can be easily compressed i.e. its volume can be easily altered, when subjected to pressure, and not only that, but if the pressure on the gas is gradually decreased, it can also be made to expand indefinitely, occupying all the space made available to it. Thus the gas occupies the entire volume of the vessel in which it is kept, and whole of the gas will escape if there be the tiniest opening in the vessel.

Here, we shall discuss only the case of liquids.

1.2. THRUST AND FLUID PRESSURE

One of the fundamental properties of a liquid in equilibrium is that it exerts a force on any surface or on all bodies in contact with it, and this force acts perpendicular to the surface and is spread over the entire area of the surface in contact. The bottom or the walls of the vessel containing the liquid may be considered as such a surface. The *total* *force* exerted by a liquid column on the whole of the area in contact with it is called **thrust**. A liquid at rest always exerts a thrust normally to the surface in contact with it. If it were not so, the reaction of the bounding surface of the container on the liquid opposite to the thrust would have a component parallel to the bounding surface which would cause the liquid to flow, since a liquid cannot withstand a tangential shear. It follows, therefore, that since the liquid is at rest, the thrust due to it must be perpendicular to the bounding surface at every point. Further, since every layer of a liquid at rest is in equilibrium, this means that the downward thrust on it, due to the liquid column above must be balanced by an equal upward thrust due to the liquid column below it. In other words, at any particular level of the liquid at rest, the downward thrust due to the liquid column above it is exactly balanced by the upward thrust on it.

If this force is uniformly distributed over the area i.e., its value is the same on each small area element of the surface, then its value per unit area is called **pressure** due to the liquid at rest or **hydrostatic pressure**. Thus, if a force F due to the liquid is acting uniformly on an area A in contact with it, then pressure

$$p=\frac{F}{A}$$

Thus, **thrust = pressure × area**

However, the force and hence the pressure may not be uniform. In such a case, the average pressure acting over the given surface is,

$$p_{av}=\frac{\delta F}{\delta A}$$

To define the pressure at a specific point, consider a small area element δA containing that point. In such a case, the small force δF acting over this area gives the pressure $\frac{\delta F}{\delta A}$.

When δA is vanishingly small, the pressure at the point is,

$$p=\lim_{\delta A\to 0}\frac{\delta F}{\delta A}$$

The SI unit of pressure is pascal (Pa), where 1Pa = 1N/m^2. Its dimensional formula is [$ML^{-1}T^{-2}$].

1.3. VARIATION OF HYDROSTATIC PRESSURE WITH DEPTH IN A LIQUID AT REST UNDER GRAVITY

Consider a liquid of density ρ which is in equilibrium of rest in a vessel containing it. Let R and Q be two points inside the liquid at a vertical distance h, and consider an imaginary cylindrical column of the liquid with axis RQ, cross-sectional area A and length or height h, such that points R and Q lie on the flat faces of the cylinder (Fig.1.1).

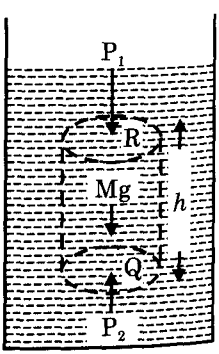


Fig: 1.1

The mass of the liquid inside the imaginary cylinder is,

$$M = \text{volume of liquid} \times \text{density} = Ah\rho$$

Let P_1 and P_2 be the hydrostatic pressure of the liquid at points R and Q respectively. The liquid cylinder is under the action of the following vertical forces:

(*i*) Force $F_1 = P_1A$, acting vertically downwards on the top face of the cylinder.

(*ii*) Force $F_2 = P_2A$, acting vertically upwards on the bottom face of the cylinder.

(*iii*) Weight $Mg = Ah\rho g$, of the liquid cylinder acting vertically downwards, where g is the acceleration due to gravity at the place.

As the liquid is in equilibrium, therefore, the liquid cylinder must also be at rest in equilibrium. Hence, the net force on it must be zero,

$$i.e.,\ F_1 + Mg - F_2 = 0$$

or $$P_1A + Ah\rho g - P_2A = 0$$

or $$P_2 - P_1 = hg \tag{1.1}$$

Hence in a liquid at rest under gravity, the pressure difference between any two points varies as the vertical distance between them.

We consider the following cases:

(*a*) If the points R and Q lie at the same level inside the liquid, then h = 0.

Therefore, from eq. (1.1), $P_2 = P_1$ at all points.

This shows that the hydrostatic pressure is the same at all points inside the liquid lying at the same horizontal level.

(*b*) If point R is shifted to the top of the liquid surface, which is exposed to the atmosphere, then $P_1 = P_a$ = atmospheric pressure. Then the total pressure at point Q at depth h inside the liquid is,

$$P - P_a = h\rho g \quad \text{or} \quad P = P_a + h\rho g \tag{1.2}$$

This shows that P is greater than P_a by an amount $h\rho g$. This excess pressure at depth h below the liquid surface is called the gauge pressure at depth h or at the point Q.

(*c*) If the acceleration due to gravity is zero i.e. g = 0, then from eq. (1.2), $P = P_a$.

This shows that there will be no difference in the hydrostatic pressure at all points inside the liquid, irrespective of the height from the free surface of the liquid, and the pressure is simply equal to the atmospheric pressure. Thus the pressure exerted by a liquid at various points inside it is due to the effect of gravity. In other words, it is due to the weight of the liquid above the layer or point under consideration.

(*d*) If P = Pressure due to the liquid column of height h at the point Q, then

$$P_2 - P_1 = P = h\rho g$$

This shows that the pressure exerted by a liquid column of height h is independent of the area of cross-section of the column, but depends only upon the height of the liquid column and the density of the liquid.

1.4. HYDROSTATIC PARADOX

From the previous case (d), it follows that, for a given liquid, so long as the vertical height of the column of a liquid remains the same, the pressure exerted by it remains the same, irrespective of its actual mass or weight. The pressure inside a liquid remains the same at all points on the same horizontal level in it.

To understand the hydrostatic paradox, consider five different vessels of different shapes as shown in fig.1.2, all having equal bases and containing water up to the same vertical height *h*.

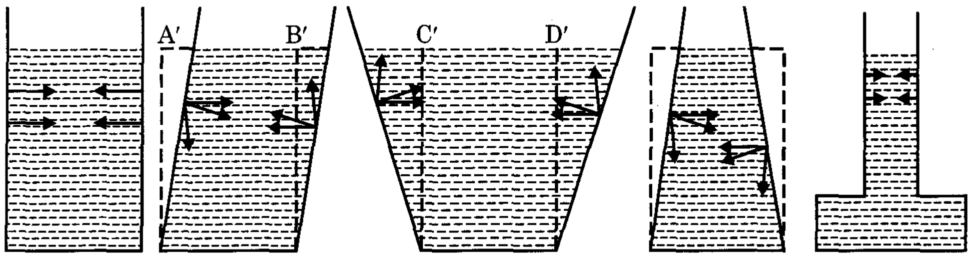


Fig: 1.2

In case of vessel A, the thrust on the base of the vessel is due to the entire weight of the cylindrical column of height *h.* In vessel B, the upward component of the normal reaction due to the left side of the vessel supports the weight of the water in between the left side and the dotted line *A′*. While the downward component of the normal reaction due to the right side of the vessel exerts a downward thrust on the base of the vessel equal to the weight of the water in between the right side and the dotted line *B′*; that the net thrust on the base is the same as due to the vertical column *h* of water. In vessel C, the upward components of the normal reaction due to both the left and right sides of the vessel support the extra weight of water in between the left and right sides and the dotted lines *C′* and *D′* respectively; so that the net thrust on the base of the vessel is equal to the cylindrical column *h* of water in between the dotted lines *C′* and *D′*. This same principle is also true for the other two vessels.

Similarly, if all the vessels are connected together with the same horizontal base (Fig. 1.3.), then if water is poured into the vessel, it will be seen that water rises to the same height in all the sections of the vessel.

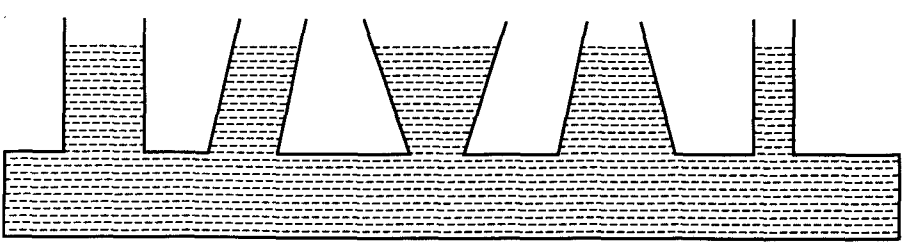


Fig: 1.3

1.5. PASCAL'S LAW

According to Pascal's law, the pressure exerted anywhere in a confined fluid at rest is transmitted undiminished equally in all directions throughout the volume of the fluid.

A simple theoretical proof of Pascal's law may be given on the basis of the law of conservation of energy. Fig.1.4. shows two cylindrical vessels B and C fitted with air-tight, frictionless pistons P_1, P_2 with cross-sectional areas A_1, A_2 respectively. These vessels are connected together by a narrow, horizontal tube D and filled with an incompressible liquid.

Let a force F_1 acting on P_1 push it downward while a force F_2 holds the piston P_2 in equilibrium. If P_1 is allowed to move downwards through a distance l_1, a Volume $A_1 l_1$ of liquid is pushed into the cylinder C from the cylinder B. Since the liquid is incompressible, the piston P_2 will be forced to move outwards through a distance l_2 (say) to accommodate the liquid flowing from cylinder B to C. Clearly,

$$A_1 l_1 = A_2 l_2 \qquad (1.3)$$

Work done by the force F_1 in moving the piston P_1 through $l_1 = F_1 l_1$

Work done by the force F_2 in moving the piston P_2 through $l_2 = F_2 l_2$

Since no energy is stored in the system, therefore

$$F_1 l_1 = F_2 l_2 \qquad (1.4)$$

From eqns. (1.3) & (1.4),

$$\frac{F_1 l_1}{A_1 l_1} = \frac{F_2 l_2}{A_2 l_2}$$

$$\Rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

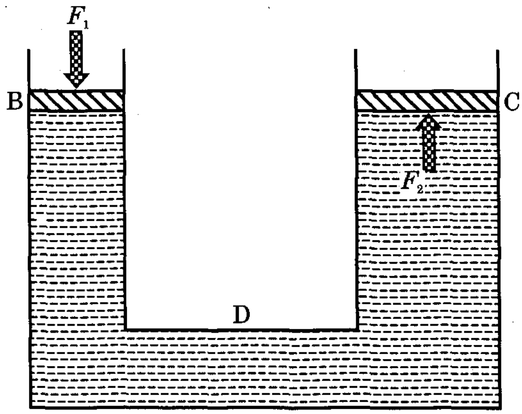


Fig: 1.4

Thus, pressure exerted by piston P_1 = Pressure exerted on piston P_2

Hence, the applied pressure is transmitted undiminished.

Pascal's law is applied in hydraulic lift, in hydraulic press and in hydraulic brakes. Pascal's law serve as the principle of working of JCB's.

1.6. ARCHIMEDES PRINCIPLE

The Greek philosopher, Archimedes was the first to discover that, whenever a body is fully or partly immersed in a liquid, it experience an upward thrust acting on it equal to the weight of the liquid displaced. Thus the body appears to be lighter inside the liquid, i.e. there is an apparent decrease in the weight of the body equal to the weight of the liquid it displaces.

Consider a homogenous body in the shape of a cylinder of length l which is completely immersed inside a liquid as shown in Fig. 1.5. The liquid exerts normal thrust on all the faces of the cylinder. The thrust exerted by the liquid on the curved surface area of the cylinder on all sides are equal and opposite and hence cancel each other's effect. Therefore, the net thrust on the liquid is only along the vertical direction along the two end faces.

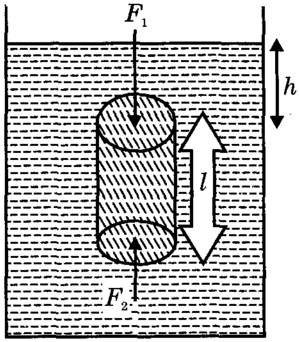


Fig: 1.5

If the top face of the cylinder is at a distance h from the free surface of the liquid, then the pressure acting on the top face is,

$$P_1 = gh\rho$$

(where ρ is the density of the liquid)

If 'A' be the face area of the cylinder, the downward thrust acting on it due to the liquid is,

$$F_1 = P_1 \times A = gh\rho A$$

The pressure on the bottom face of the liquid is, $P_2 = g(h + l)\rho$, so that the upward thrust acting on it due to the liquid is,

$$F_2 = P_2 \times A = g(h + l)\rho A$$

Therefore, the net upward thrust acting on the liquid $= F_2 - F_1$

$= Al\rho g$

= Volume of liquid displaced × density of liquid × acceleration due to gravity

= Weight of liquid displaced

Thus, when a body is wholly or partially immersed in a liquid, it experiences an upthrust equal to the weight of the liquid displaced by it. This is known as Archimedes principle.

The resultant upward thrust i.e. uphrust, is called the force of buoyancy, buoyant force or simply buoyancy. It acts vertically upwards through a point, which is the centre of gravity of the displaced liquid and is called the ***centre of buoyancy*** or centre of floatation.

In case of a homogenous body the centre of buoyancy coincides with the centre of gravity of the body. Since the *true weight* (W) of the body acts vertically downwards and the buoyant force (F_b) acts vertically upwards, and is equal to the weight of the displaced liquid (W'), therefore, the *apparent weight* of the body inside the liquid is

$$W_{app} = W - F_b = W - W' \qquad (1.5)$$

Thus, the body appears lighter when it is immersed inside a liquid and the loss in weight is equal to the weight of the liquid displaced.

If σ be the density of the solid body which is immersed inside the liquid, then from eq. (1.5), we have

$$W_{app} = Al\,\sigma\,g - Al\,\rho\,g = Al\,(\sigma - \rho)g = Al\,\sigma\,g\left(1-\frac{\rho}{\sigma}\right)$$

$$\text{or} \qquad W_{app} = W\left(1-\frac{\rho}{\sigma}\right)$$

$$\therefore \qquad \text{The true weight of the solid, } W = \frac{W_{app}}{\left(1-\frac{\rho}{\sigma}\right)} \qquad (1.6)$$

From the above equation, if:

(*i*) $\sigma < \rho$ *i.e.*, the density of the solid is less than the density of the liquid, then the true weight of the solid will be less than its apparent weight which means that the upthrust on the solid is greater than its weight. Then the solid will rise to the surface of the liquid up to the extent that the weight of the displaced liquid (*i.e.*, the upthrust) is equal to the weight of the solid immersed in it and the solid will float on the surface.

(*ii*) $\sigma > \rho$, then the true weight of the solid will be greater than its apparent weight *i.e.*, the upthrust on it is less than its weight. In this case the solid will sink to the bottom of the liquid.

(*iii*) $\sigma = \rho$, then the true weight of the solid will be equal to its apparent weight *i.e.*, the upthrust due to the liquid is equal to the weight of the solid. In this case the solid will be at rest anywhere inside the liquid. If the whole volume of the solid is immersed inside the liquid, then it will float inside the liquid.

1.7. SURFACE PHENOMENA IN LIQUIDS: SURFACE TENSION

Surface tension is essentially a molecular phenomenon and hence, it is important to first of all learn and have a clear idea about the forces that operate between the molecules of substances. The forces between the molecules of a substance are called the intermolecular forces. The intermolecular forces are of electrical origin and these forces are known as Vander Waal's forces. These forces are attractive in nature but are different from the ordinary gravitational forces, and do not obey inverse square law. The molecular forces are found to vary inversely as the seventh power of distance i.e., their magnitude increases rapidly with decrease in the distance between the molecules.

There are two types of molecular forces: (*i*) Forces of adhesion or adhesive forces and (*ii*) Forces of cohesion or cohesive forces.

(*i*) *Adhesion is the force of attraction acting between the molecules of different substances*, and is different for different pairs of substances.

For example, the adhesive force between the molecules of water and glass makes the water to wet the glass. Similarly, while writing, graphite from pencil sticks to the paper due to the adhesive force. On the other hand gum has a greater adhesive force for solid surfaces than liquids.

(*ii*) *Cohesion, on the other hand, is the force of attraction acting between the molecules of the same substance.*

For example, Solids have a definite shape due to the strong forces of cohesion between the molecules of a solid. Liquids have a definite free surface but no definite shape which shows that the cohesive force is weaker in liquids than in solids, while gases have neither a definite shape nor a definite free surface, which shows that the cohesive force is the weakest in gases. Mercury does not wet the walls of the glass container because the cohesive force between the molecules of mercury is stronger than the adhesive force between the molecules of mercury and glass.

1.8. MOLECULAR RANGE – SPHERE OF INFLUENCE

The maximum distance up to which the force of cohesion between two molecules can act is called their *molecular range.* It represents the maximum distance up to which a molecule can exert a measurable attraction on other molecules, and is generally of the order of 10^{-9}m in the case of solids and liquids, being different for different substances.

A sphere drawn with a molecule as centre and radius equal to its molecular range is called the sphere of influence of the molecule. All the molecules lying inside this sphere attract the molecule at the centre and are also attracted by the molecule, whereas the molecules lying outside the sphere do not affect and are not affected by the molecule, at the centre.

The topmost layer of a liquid at rest with thickness equal to molecular range is called the *surface film.*

1.9. SURFACE TENSION: MEASUREMENT OF

Whenever a body is stretched it is in a state of tension, and because of its elastic properties it has a natural tendency to contract. For example, when we stretch a rubber tube it has a tendency to shorten its length and if a rubber sheet is stretched it has a tendency to reduce its area. Similarly, on account of the *cohesive forces* between the molecules of a liquid, the free surface of a liquid always behaves like a *stretched elastic membrane and hence has a natural tendency to contract and assume the smallest possible area.* This tendency of a liquid to decrease its surface area is illustrated by the following examples.

(*i*) It is a general experience that a liquid in small quantity at rest, free from external forces like gravity, always tends to assume the shape of a spherical drop *e.g.*, rain drops, small quantities of mercury placed on a clean glass plate, a freely suspended drop of water form at the end of a tap etc. Since, for a given volume, a sphere has the least surface area, the liquid assumes a spherical shape in order to have minimum possible surface area. Ordinarily the effect is not as marked, as the liquids tend to spread due to the force of gravity. If the force of gravity is eliminated the liquid will assume a perfectly spherical shape.

(*ii*) When a shaving brush is dipped in water, its hairs spread out. On taking out the brush the hairs of the shaving brush are pressed together. This is because the water film formed between the hairs while tending to make its surface area minimum due to surface tension will bring the hairs closer to each other.

(*iii*) If a greased needle be placed on a piece of blotting paper and the latter is gently placed on the surface of water at rest, the blotting paper will soon sink to the bottom, but the needle will remain floating on the surface. A careful observation shows that the surface of water below the needle gets slightly depressed. This shows that the free surface of water supports the needle in the same way as a stretched rubber membrane does. If the layer is pricked by one end of the needle, it sinks down.

(*iv*) Some insects, like mosquitoes etc, can move about freely on the surface of ponds without sinking. This is on account of the support they get from the free surface of water which behaves like a stretched membrane.

Measurement of surface tension: Imagine a line *AB* drawn tangentially anywhere on the free surface of a liquid; the molecules lying just on its one side tries to pull it away from the molecules lying just on the other side in order to decrease the surface area. Thus the force of surface tension acts at right angles to this line on both sides and tangentially to the liquid surface (Fig.1.6.). The force acting per unit length of such a line gives a quantitative measure of surface tension.

Hence surface tension is defined and measured as the force per unit length acting on either side of an imaginary line drawn tangentially anywhere on the liquid surface in equilibrium, the direction of the force being tangential to the surface and perpendicular to the line.

Let F be the total force acting on either side of an imaginary line of length l, drawn tangentially on the liquid surface at rest. The force of surface tension T, by definition, is given by

$$T = \frac{F}{l}$$

T is constant for a given liquid (free from impurities) at a given temperature. It is measured in Newton per meter (Nm^{-1}) in S.I units.

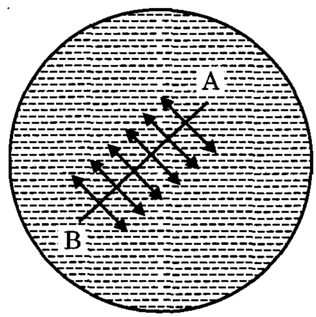


Fig: 1.6

1.9. MOLECULAR THEORY OF SURFACE TENSION

The contractile tendency of a liquid surface at rest has been explained by Laplace on the basis of the molecular theory, discussed below:

A molecule attracts and is in turn attracted by the molecules lying within its *sphere of influence*. As shown in Fig. 1.7., let PQRS be the surface film of a liquid at rest in a container, where PS = QR = molecular range. Consider three molecules of the liquid; *A* well inside it, *B* just below the surface and *C* on the surface, with their sphere of influence drawn around them. Since the sphere of influence of molecule *A* lies completely inside the liquid, it is equally attracted in all directions by the molecules lying within its sphere of influence. Hence there is no resultant

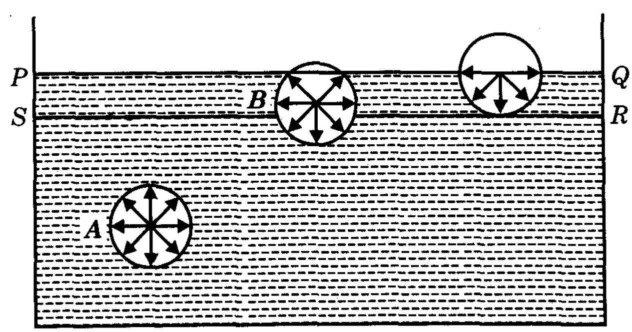


Fig: 1.7

cohesive force acting on it.

The sphere of influence of molecule B, on the other hand, lies partly outside the liquid. Therefore the number of molecules on the upper half of the molecule B pulling it upwards is less than the number of molecules on its lower half pulling it downwards. Therefore, there is a resultant downward force of cohesion acting on the molecule B.

Since the sphere of influence of molecule C lies exactly half inside and half outside, hence the number of molecules in the lower half of sphere of influence of molecule C, attracting it downwards, is very large as compared to the number of molecules (only gas or vapour molecules) in the upper half of its sphere of influence attracting it upwards. Therefore there is maximum resultant downward force acting on the molecule C.

Clearly this is true for all the molecules of the liquid. Hence all the molecules in the surface film are pulled downwards due to the resultant cohesive force, the magnitude of which increases from the bottom (SR) of the surface film to its top (PQ) i.e. the free surface of the liquid. Now, if a molecule from the interior of the liquid is brought up to the surface film, work has to be done against the downward cohesive force on it and hence its potential energy increases. Therefore the molecules in the surface film have more potential energy than the molecules lying below it. The greater the number of molecules in the surface film, the larger is the potential energy of the surface film. A system in equilibrium tends to acquire minimum possible potential energy. Thus, in order to attain a stable equilibrium the surface film also tends to have minimum potential energy, which will be so if the number of molecules in the surface film is minimum. For this the surface film should have minimum volume. This can be done only by decreasing the surface area of the film, because its thickness is already fixed (equal to molecular range). Thus the free surface of a liquid at rest always tends to have minimum surface area, and in so doing the surface film tends to contract and behaves like a stretched membrane.

1.11. SURFACE ENERGY

It is seen from above that the potential energy of molecules in the surface film is greater than those in the interior of the liquid. The excess of potential energy per unit area of the film is called its *surface energy*. Also the free surface of a liquid at rest tends to contract so as to acquire minimum surface area due to surface tension. Therefore, if the free surface of the liquid is to be increased, work has to be done against the force of surface tension. This work done is stored in the surface film as part of *surface energy* of the increased surface area.

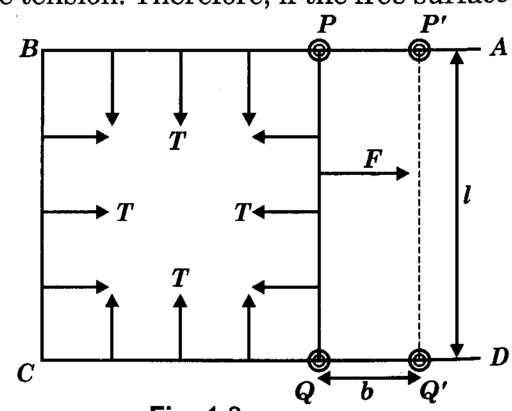


Fig: 1.8

Consider a rectangular wire frame $ABCD$, with a horizontal wire PQ in it which can slide freely on the sides AB and CD (Fig.1.8.). Let a soap film be formed over it. The wire PQ is pulled inwards due to surface tension, acting perpendicularly to the wire and in the plane of the film, by a force $2T \times l$, where T is the surface tension of the soap film and l is the length of the wire PQ. The factor '2' appears because there are two surfaces of the soap film. If the wire PQ is pulled outward by a small distance b to the position $P'Q'$, keeping the temperature constant, then

The work done by the external force $= 2T \times l \times b = 2Tlb$

Increase in area of the soap film $= 2 \times l \times b = 2lb$

$\therefore$ Work done or Energy spent per unit area $= \dfrac{2Tlb}{2lb} = T$

This energy is stored in the increased surface area of the soap film.

Hence the surface tension is numerically equal to the work done in increasing the surface area of liquid film by unity under isothermal conditions. In other words, the surface energy per unit area of a surface film is numerically equal to the surface tension.

Thus, surface tension can be expressed in joules per square metre (Jm^{-2}).

In the above discussion, it is assumed that the temperature of the film remains constant. But in the actual case, the film gets cooled on being stretched, because the drawing out of the molecules from the interior of the liquid to the surface results in the increase of the potential energy of the molecules. This increase in potential energy of the molecules is at the expense of kinetic energy of the molecules. This leads to a decrease in their thermal agitation, with a consequent lowering of temperature. Thus the film gets cooled and it therefore, absorbs heat from the atmosphere to keep its temperature constant. This heat absorbed, together with the mechanical work done, forms the energy of the new surface area $2lb$ of the film formed.

Thus, if E is the surface energy of the film and H is the amount of heat absorbed per unit area of the new surface formed, we have

$$E \times 2lb = 2lb \times T + H \times 2lb$$

or $$E = T + H$$

$\Rightarrow$ $$T = E - H = \text{(Surface energy – Heat energy per unit area)}$$
$$= \text{Potential energy per unit area}$$

i.e., $$T = \text{Work done in forming unit area of the surface film}$$

Thus, Surface tension is equal to the mechanical part of the surface energy of the liquid film, which may be called as *free surface energy* of the liquid-film or surface.

1.12. PRESSURE DIFFERENCE ACROSS A CURVED LIQUID SURFACE

All the molecules lying within the surface film of a liquid are pulled downwards due to the resultant downward cohesive force between the molecules of the liquid. *This downward force exerted per unit area of a liquid surface is called cohesive pressure.*

When the free surface of a liquid is *plane,* a molecule in the surface is attracted by other liquid molecules equally in all directions. Consequently, the resultant force on a molecule due to surface tension is *zero* as shown in fig. 1.9. (*i*), and the cohesion pressure is negligible. But if the liquid surface is curved, there is a resultant force of surface tension which acts normally to the surface.

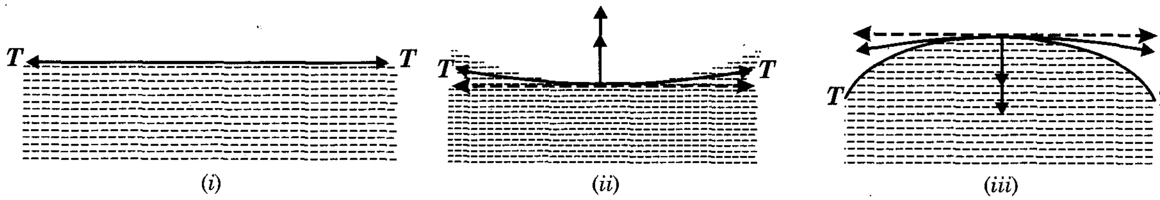


Fig: 1.9

If the free surface of a liquid is *concave*, the resultant force on it acts *outwards* (away from the liquid) as shown in fig. 1.9.(*ii*) and the cohesion pressure is *decreased*. But if the free surface of a liquid is *convex*, the resultant force on it acts *inwards* (into the liquid) as shown in fig. 1.9.(*iii*) and the cohesion pressure is *increased*. Hence, for the equilibrium of a curved surface, there must be a pressure difference across it so that the force of surface tension will be balanced by the excess pressure acting on the concave side of the liquid surface.

1.13. EXPRESSION FOR EXCESS PRESSURE ON A CURVED LIQUID SURFACE.

From the above section it is seen that if we have a curved liquid surface at rest, then the inward/outward pressure on it due to surface tension must be balanced by an equal excess of pressure, acting on the concave side of the curved liquid surface.

Let ABCD be a small cuvilinear element of the curved liquid surface at rest as shown in Fig. 1.10. Let the length of side AB be x and radius of curvature r_1 with centre at O_1, and the side BC is of length y and radius of curvature r_2 with centre at O_2. Clearly, AO_1, $= BO_1 = r_1$ and $BO_2 = CO_2 = r_2$. Geometrically radii of curvature, such as AO_1 and BO_2 are called principal radii of curvature. Surface area of the curvilinear element $ABCD = xy$.

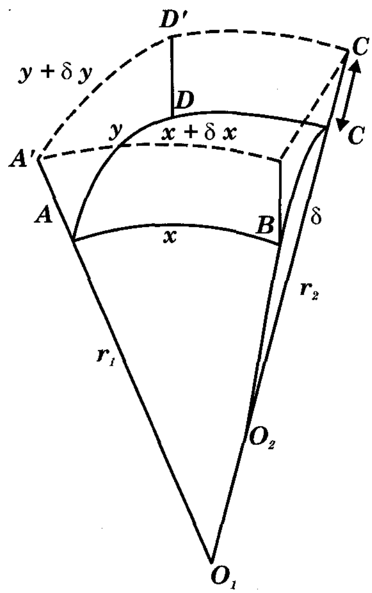


Fig: 1.10

Let the excess of pressure on the concave side be p. When the liquid surface is at rest, the outward force on the element ABCD due to excess pressure $= p \cdot xy$

Suppose the surface is expanded by giving an infinitesimally small normal displacement δz ($= A A'$) so that the element occupies a new position $A' B' C' D'$, its curvature remaining unchanged. Hence, the work done by the excess pressure p is

$$W = \text{force} \times \text{displacement}$$

$$= p \cdot xy \times \delta z \qquad (1.8)$$

If $x + \delta x$ and $y + \delta y$ be the lengths of $A' B'$ and $C' D'$ respectively, then increase in the area of the liquid surface under consideration

$$= (x + \delta x)(y + \delta y) - xy$$

$$= x\,\delta y + y\,\delta x \text{ (the product } \delta x\ \delta y \text{ of very small quantities has been neglected).}$$

$\therefore$ Increase in the surface energy of the element =
surface tension × increase in area = $T\,(x\,\delta y + y\,\delta x)$ (1.9)

Where, T is the surface tension of the liquid. This increase in the surface energy is equal to the work done in expanding the surface by excess of pressure, hence equating (1.8) and (1.9), we get

$$pxy\,\delta z = T(x\,\delta y + y\,\delta x)$$

or $$p = T\left(\frac{1}{y}.\frac{\delta y}{\delta z} + \frac{1}{x}.\frac{\delta x}{\delta z}\right) \tag{1.10}$$

Now from similar traingles ABO_1 and $A'\,B'\,O_1$, we have

$$\frac{A'B'}{A'O_1} = \frac{AB}{AO_1} \quad \text{or} \quad \frac{x+\delta x}{r_1+\delta z} = \frac{x}{r_1}$$

$[\because A'O_1 = AO_1 + AA' = r_1 + \delta z]$

or $$\frac{x+\delta x}{x} = \frac{r_1+\delta z}{r_1} \quad \text{or} \quad 1+\frac{\delta x}{x} = 1+\frac{\delta z}{r_1}$$

or $$\frac{\delta x}{x} = \frac{\delta z}{r_1} \Rightarrow \therefore \frac{1}{x}.\frac{\delta x}{\delta z} = \frac{1}{r_1}.$$

Similarly, from similar triangles BCO_2 and $B'\,C'\,O_2$, we have

$$\frac{1}{y}.\frac{\delta y}{\delta z} = \frac{1}{r_2}$$

Substituting these values of $\frac{1}{x}.\frac{\delta x}{\delta z}$ and $\frac{1}{y}.\frac{\delta y}{\delta z}$ in eq. (1.10), we get

$$p = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \tag{1.11}$$

If one of the curvatures is *convex* and the other is *concave*, the radii of curvature r_1 and r_2 are of *opposite signs*. Thus, in such case, we have

$$p = T\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{1.12}$$

Combining the equation (1.11) and (1.12), we may write the general realtion as

$$p = T\left(\frac{1}{r_1} \pm \frac{1}{r_2}\right) \tag{1.13}$$

If instead of a liquid surface, there are two surfaces of a film or membrane then the excess of pressure is given by

$$p = 2T\left(\frac{1}{r_1} \pm \frac{1}{r_2}\right) \quad (1.14)$$

The relations (1.13) and (1.14) hold for all surfaces. A few important particular cases are given below.

(*i*) **Spherical surface.**

(*a*) In case of a single spherical surface such as that of a liquid drop or air bubble inside a liquid, we have $r_1 = r_2 = r$ (say). Hence from (1.13)

$$p = T\left(\frac{1}{r} + \frac{1}{r}\right) = \frac{2T}{r} \quad (1.15)$$

(*b*) In case of a single spherical surface such as that of a liquid drop or air bubble inside a liquid, we have $r_1 = r_2 = r$ (say). Hence from (1.14)

$$p = 2 \times \frac{2T}{r} = \frac{4T}{r} \quad (1.16)$$

(*ii*) **Cylindrical surface.**

(*a*) Here one of the radii is the same as the radius (r) of the cylinder while the other is infinite *i.e.*, $r_1 =$ (say) and $r_2 = \infty$. For one single surface, such as for a cylindrical column of liquid or for a cylindrical bubble in a liquid, from (1.13) we obtain

$$p = T\left(\frac{1}{r} + \frac{1}{\infty}\right) = \frac{T}{r} \quad (1.17)$$

For two surfaces such as for a cylindrical bubble or film, we get from (1.14)

$$p = 2T/r.$$

(*iii*) **Case of a catenoid.**

If the surface is one of revolution with no difference of pressure, the surface is a catenoid. Here p = 0, hence we have

$$\frac{1}{r_1} \pm \frac{1}{r_2} = 0 \quad (1.18)$$

An example of such a surface is that of a film, supported in between two parallel rings and ends of the film are burst so that p = 0, or, better still, between the mouths of two funnels, obtained by dipping them in a soap solution with their mouths in contact and then slowly pulling them apart, with their narrower ends open to the atmosphere.

1.14. EXCESS PRESSURE INSIDE A LIQUID DROP

A liquid drop is spherical in shape and hence the surface of a liquid drop is convex and, therefore, the molecules on the surface experience a resultant force due to surface tension, acting inwards. Therefore for the equilibrium of the spherical drop, the resultant inward force must be balanced by the excess pressure, acting on the concave side of the spherical surface.

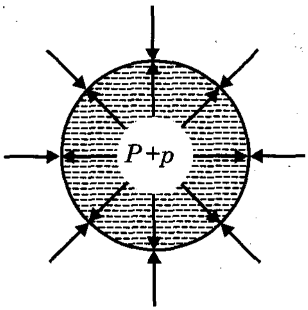


Fig: 1.11 (a)

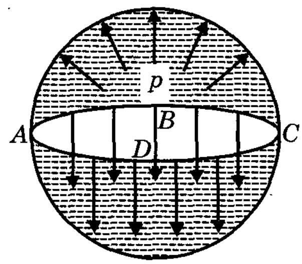


Fig: 1.11 (b)

Let r be the radius of the drop and P be the pressure on the convex side and $P + p$ that on its concave side [Fig.1.11 (a)]. Therefore excess of pressure inside the drop is p.

Now let us consider the equilibrium of one half of the drop, as shown in the fig.1.11 (b). Neglecting the weight of the drop, there are two forces acting on it:

(*i*) The outward force on the plane face *ABCD*, due to excess of pressure p,

$$= \text{excess of pressure} \times \text{area of plane face } ABCD$$

$$= p \times \pi r^2$$

(*ii*) The inward force, due to surface tension, acting on the rim *ABCD* of the hemispherical drop,

$$= \text{surface tension of the liquid} \times \text{circumference of the circle } ABCD$$

$$= T \times 2\pi r$$

Since the hemispherical drop is in equilibrium hence the outward force must be equal to inward force,

$$\text{i.e.} \quad p \times \pi r^2 = T \times 2\pi r, \qquad \therefore p = \frac{2T}{r} \tag{1.19}$$

1.15. EXCESS PRESSURE INSIDE AN AIR BUBBLE IN A LIQUID

An air bubble inside a liquid will have only one spherical surface. There will be the excess of pressure p on the concave side, i.e., inside the bubble and hence this case is dealt exactly in the same manner as described above [Art.1.14]. The expression for the excess of pressure, as derived above, is

$$p = \frac{2T}{r} \tag{1.20}$$

1.16. EXCESS PRESSURE INSIDE A SOAP BUBBLE

A spherical soap bubble has two free surfaces in contact with air. There is an excess of pressure acting on the concave side of the bubble as shown in Fig1.12 (a). If p be the excess of pressure then by considering the equilibrium of one half of the spherical bubble [Fig.1.12(b)], the outward force on the plane face $ABCD$ of the hemisphere due to the excess pressure $= p \times \pi r^2$.

The inward force of surface tension on the circumference of the circle $ABCD$ due to a single surface is $T \times 2\pi r$, hence the total force of surface tension due to both the surfaces is

$$= 2 \times T \times 2\pi r$$

For the equilibrium of the bubble, the outward force due to excess of pressure must be equal to the inward force due to surface tension. Hence,

$$p \times \pi r^2 = 2 \times T \times 2\pi r, \qquad \therefore p = 4T/r. \qquad (1.21)$$

Thus we see that the excess pressure inside a drop or bubble is inversely proportional to its radius, so that the smaller the bubble or drop, the greater is the excess pressure inside it.

1.17. DETERMINATION OF SURFACE TENSION OF LIQUID — JAEGER'S METHOD

We know that the excess pressure inside an air bubble in a liquid is equal to $2T/r$ [Art.1.15], where T is the surface tension of the liquid and r, the radius of the bubble. Jaeger measured this excess pressure p, necessary to produce such a bubble, by a simple method described below. So that, knowing p and r, the surface tension (T) of the liquid can be determined.

The apparatus, used for this purpose, consists of a long thin glass tube AB, with its lower portion ending in a fine jet of about 0.2 to 0.5 mm in diameter, and its tip is cut perpendicular to the axis of the tube and quite smooth so that even under a microscope, there appears no trace of roughness or ruggedness at its inner or outer edges. This tube is dipped in the experimental liquid, contained in a vessel, with about 4 to 5 cm of its length inside the liquid,— the diameter of the vessel being at least 8cm to ensure the flatness of the liquid surface in it. It is then connected to a manometer M and a Woulff's bottle W, fitted with a dropping funnel F containing (water or mercury) as shown in Fig.1.13. The liquid used in the manometer is usually of low density (such as Xylol), so that the difference in the levels of the liquid in the in its limbs may be large for a given pressure difference.

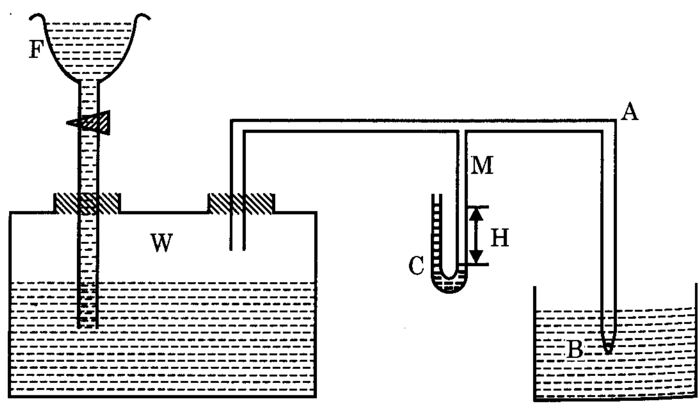


Fig: 1.13

Due to capillary action, some liquid rises up into the tube AB to a height well above the level of the liquid outside, the shape of its meniscus being nearly hemispherical. Some air is then forced into the tube by dropping water from the Woulff's bottle, which displaces its own volume of air from it. The liquid column in AB slowly moves down until it reaches the end B and then a bubble is formed there. As the pressure inside the bubble increases, the radius of curvature of the bubble gradually decreases, until it reaches a minimum value, and the bubble acquires a more or less hemispherical shape, with radius r, equal to that of the orifice at B, the pressure inside being now the maximum. The maximum pressure is noted from the manometer by noting the difference of levels (H) in the two limbs of the manometer. The bubble now becomes unstable; for any further growth of it due to the force of air, tends to increase its radius. This results in the decrease in the pressure inside it due to surface tension, thus destroying the equilibrium between its internal pressure and the constant external pressure. It therefore, breaks away from the tube and the whole process starts all over again.

The whole operation is so regulated that, (*i*) only one bubble is formed at a time and (*ii*) it takes about 10 seconds for one bubble to be formed at B.

Just before the bubble breaks away from B, the pressure inside it is equal to that at C, *i.e.*, $P + H\,\rho\, g$, where P is the atmospheric pressure and $H\,\rho\, g$ is the pressure due to the liquid column H in the manometer, and ρ is its density. When the bubble just breaks away from B, the pressure on it is equal to that at the level of B in the beaker, *i.e.*, $P + h\sigma g$, where h is the depth of orifice B below the surface of the experimental liquid and σ, its density.

$$\text{Therefore, the excess pressure inside the bubble} = (P + H\,\rho\, g) - (P + h\,\sigma\, g)$$

$$= (H\rho - h\sigma)g$$

$$\text{But as the excess pressure inside the bubble} = 2T/r$$

$$\therefore\ 2T/r = (H\rho - h\sigma)g$$

$$T = \frac{(H\rho - h\sigma)gr}{2}$$

Thus, knowing H, h, ρ and σ, and determining r with the help of a microscope fitted with a micrometer eye-piece, the value of surface tension T may be determined.

The method is not very accurate in so far as the determination of absolute values of surface tension is concerned, as the phenomenon is a dynamical one and the exact value of the radius of the bubble at the point of breaking is uncertain. This method however, can be used to study the variation of surface tension with temperature, since the temperature of liquid in the containing vessel can be easily controlled and the bubble is formed inside the liquid itself. The method is also applicable for comparison of surface tensions of different liquids, determination of surface tensions of molten metal, and studying the variation of surface tension of a solution at different concentrations of solutes.

Note: To avoid the quantity σ, the density of the test liquid, two tubes of radii r_1 and r_2 may be taken. If H_1, H_2 be the maximum readings in the manometer in the two cases, then

$$2T/r_2 = (H_2\,\rho - h\sigma)\,g \qquad \& \qquad 2T/r_1 = (H_1\,\rho - h\sigma)g$$

Subtracting the above two equations, we have

$$2T\left(\frac{r_2 - r_1}{r_1 r_2}\right) = g\,\rho\,(h_1 - h_2)$$

$$\therefore \qquad T = \frac{g\rho r_1 r_2 (h_1 - h_2)}{2(r_2 - r_1)}$$

This is known as Sugden's modification. Sugden also gave a more precise theory over Jaeger's allowing for the fact that the bubbles formed are not exactly spherical in shape.

1.17. SHAPE OF LIQUID MENISCUS IN A CAPILLARY TUBE

A narrow tube with a very fine bore is called a capillary tube. When a liquid is brought in contact with a solid, its surface becomes curved near the point of contact, thus making the shape of the liquid meniscus inside the capillary tube to be concave or convex. The nature of the curvature depends on the relative magnitudes of the force of cohesion between the molecules of the liquid and the force of adhesion between the molecules of the liquid and that of the solid.

Let a capillary tube of glass be dipped vertically in a liquid, meeting its surface at P (Fig. 1.14.). Then a liquid molecule at P will be acted upon simultaneously by two forces (neglecting the weight of the liquid). These are:

(*i*) A resultant adhesive force due to the molecules of the glass tube near about it and acting outwards perpendicular *i.e.*, 90°, to the glass tube at point P. Let this force be represented in magnitude and direction by PQ.

(*ii*) A resultant cohesive force due to the liquid molecules near about it and acting inwards at an angle of 45° to the vertical at P. Let this force be represented in magnitude and direction by PS.

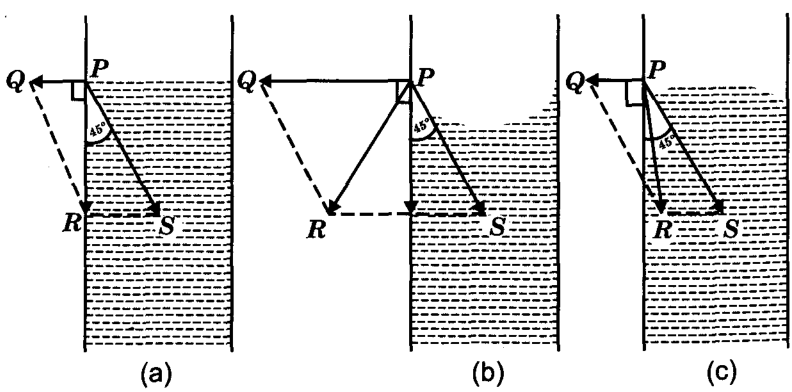


Fig: 1.14

Thus the two forces are acting at an angle of 135° with each other. The resultant of these force is obtained by the parallelogram law of vector addition, and is given by the diagonal PR of the parallelogram PQRS. The direction of the resultant depends upon the relative magnitudes of the two forces PQ and PS. The following cases may arise depending upon the nature of the liquid and the solid glass tube.

(*a*) If PQ/PS be equal to $\frac{1}{\sqrt{2}}$ *i.e.*, if PS = $\sqrt{2}$ PQ, the resultant PR will lie along the vertical, as shown in Fig. 1.14(a).

(*b*) If PS be smaller than $\sqrt{2}$ PQ, the resultant PR will lie outside the liquid, as shown in Fig. 1.14(b).

(*c*) If PS be greater than $\sqrt{2}$ PQ, the resultant PR will lie inside the liquid, as shown in Fig. 1.14(c).

All the molecules of the liquid which are similarly situated as at P lie on a circle, and experience similar resultant forces. For molecules lying away from the surface of the wall, the resultant force is inclined less and less with the vertical direction, because the magnitude of the cohesive forces increases while that of the adhesive forces decreases, and the former becomes more and more vertical as the distance from the wall increases. Since a liquid cannot permanently withstand a shearing stress, therefore, in equilibrium its surface at every point in contact with the solid will set itself at right angles to the resultant force at the point considered.

Thus in case (a), when the resultant force PR acts along the vertical, i.e. when the cohesive force is $\sqrt{2}$ times the adhesive force, the molecules of the liquid near the walls of the tube are raised up against the tube, those at the middle remaining parctically unaffected, thus making the liquid surface concave upwards. This is the cae of water or other liquids which wet the walls of the glass tube.

And, in case (c), when PR lies inside the liquid, *i.e.*, when the cohesive force is greater than $\sqrt{2}$ times the adhesive force, the liquid molecules near the walls of the tube are depressed, while those in the middle of the tube are practically unaffected, thus making the surface of the liquid convex upwards. This is the case of mercury and other liquids which do not wet the walls of the glass tube.

1.18. ANGLE OF CONTACT

In general, when the free surface of a liquid is in contact with a solid, it becomes curved near its plane of contact with the solid. ***The angle between the tangent to the liquid surface at the point of contact and the solid surface inside the liquid is called the angle of contact for the particular pair of solid and liquid in contact.***

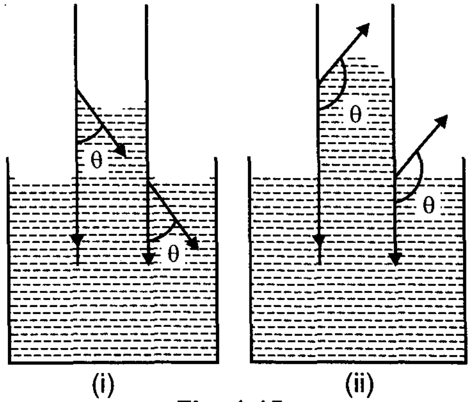


Fig: 1.15

The angle of contact depends upon the nature of the liquid and the solid in contact and is not altered by a change in the inclination of the solid inside the liquid. This angle may have any value between 0° and 180°. For example, for ordinary water and glass, the angle of contact is about 18° while for mercury and glass, it is about 138°. For most liquids and glass, however, it is less than 90°.

In general, for liquids which wet the solid surface, the liquid have a concave meniscus and the angle of contact is acute, while for liquids which do not wet the solid, it is obtuse and the liquid have a convex meniscus [Fig. 1.15(*i*) & (*ii*)]. Apart from the nature of the solid and the liquid, the angle of contact also depends upon the following factors:

(*i*) The medium which exists above the free surface of the liquid. e.g., when the angle of contact between mercury and glass is different, when a layer of air exists above the surface of mercury as when a layer of water exist.

(*ii*) The cleanness and freshness of the given two surfaces in contact.
e.g., the angle of contact for pure water and glass is 0° but if the surface of the glass be contaminated with grease, its value may be as much as 35°.

1.20. THREE MEDIA IN CONTACT:— EQUILIBRIUM OF A LIQUID DROP

I. Case of a Liquid in contact with a Solid and with air: Consider a liquid which is in equilibrium on the horizontal surface of a solid. Both the solid and the liquid are in contact with air; the common line of their contact is through P, perpendicular to the plane of the paper. Surface tension forces acts at the surface boundary. Let T_1, T_2 and T_3 be the surface tensions for liquid-air, solid-air and liquid-solid interface respectively as shown in Fig. 1.16(*a*). Let θ be the angle of contact of the liquid with the solid. Since the system is in equilibrium,

$$T_1 \cos\theta + T_3 = T_2$$

$$\Rightarrow \qquad \cos\theta = \frac{T_2 - T_3}{T_1}$$

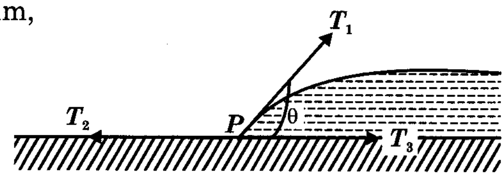


Fig: 1.16(a)

Thus, if $T_2 > T_3$, $\cos\theta$ is positive and hence θ is less than 90°. [illegible] T_3, $\cos\theta$ is negative and hence θ lies between 90° and 180°. This is the ca[illegible]rcury on glass, where T_3 is large and θ is about 138°. Therefore, mercury collects itself into drops when placed on a clean glass plate. If in case, $T_2 > T_1 + T_3$, there will be no equilibrium and the liquid will spread over the solid surface. This is the case of pure water when placed over a perfectly clean plate of glass.

II. Case of two Liquids in contact with a Solid and with air: If two immiscible liquids are brought into contact with each other at point O as shown in Fig. 1.16(b), both being in contact with air, there are three surface tensions to consider, *viz.* (a) T_1 at the surface of contact between air and liquid I, (b) T_2 at the surface of contact between air and liquid II and (c) T_3 at the surface of contact between the liquids.

For equilibrium, the three forces T_1, T_2 and T_3 must be such that the sum of any two must be greater than the third. Thus for equilibrium, the three forces must be represented by the three sides of a

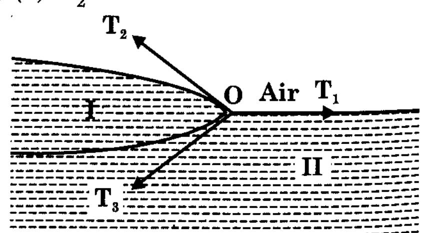


Fig: 1.16(b)

triangle, taken in order. This triangle of forces is known as the Neumann's triangle. In actual practice, no two pure liquids are known for which Neumann's triangle may be constructed, one of the surface tensions being always greater than the other two; so that, the equilibrium condition shown in the figure is never attained. Thus, for example, in the case of water, mercury and air, the water drops when placed on mercury spreads all over its surface, provided both water and mercury are pure. This is because the surface tension of mercury of mercury is about 0.55Nm-1, and that of water, only 0.075Nm-1.

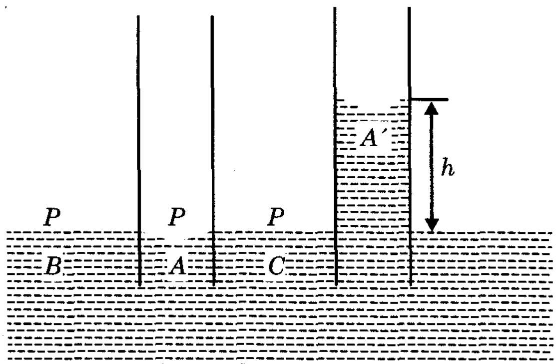
If, however, mercury surface is contaminated with grease, its surface tension decreases and some water drops may stay upon it. Thus, now the Neumann's triangle can be constructed.

1.21. CAPILLARY ACTION OR CAPILLARITY

When a capillary tube is dipped vertically in a liquid, the liquid inside it either rises or falls due to surface tension effects. This phenomenon is known as capillary action or capillarity. For example, when a capillary tube of glass is dipped in water, the water level rises inside the capillary tube. On the other hand, when the capillary tube is dipped in mercury, the level of mercury in the tube is depressed than that outside it.

1.22. CAPILLARY RISE: ASCENT FORMULA

Consider a uniform capillary tube, open at both ends, which is dipped vertically in a liquid of density ρ, which wet the walls of the tube. The angle of contact $\theta < 90^0$, and the surface of the liquid inside the tube is concave upwards i.e. the liquid meniscus is concave upwards, as shown in Fig. 1.17(a). Since the capillary tube is open to the atmosphere, the pressure over the curved surface is everywhere atmospheric. Hence the pressure at point A just below the curved surface of the liquid will be less than atmospheric by an amount $2T/R$, *i.e.*, the pressure at the point A will be $P - 2T/R$, where P is the atmospheric pressure, R is the radius of curvature of the curved surface and T is the surface tension of the liquid. The pressure outside the tube is atmospheric at all points (such as B and C) lying on the same horizontal level as A.



(a) (b)

Fig: 1.17

Due to this difference of pressure at the same horizontal level, the liquid rises up the capillary tube to a height h. Equilibrium is reached when the pressure due to the liquid column *i.e.* $h \rho g$, inside the liquid column equals the deficit of pressure at the point A i.e. $2T/R$ [Fig. 1.17(b)]. Thus,

$$2T/R = h \rho g \tag{1.22}$$

If r is the radius of the tube, then clearly from Fig. 1.17(c), we have

$$2T/R = h\rho g$$

$$\cos\theta = \frac{MN}{MO} = \frac{r}{R} \quad or \quad R = \frac{r}{\cos\theta} \qquad (1.23)$$

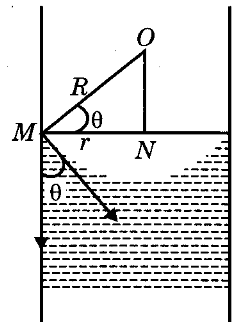


Fig: 1.17 (c)

Hence from (1.22), we get

$$2T\cos\theta = h\rho g \quad or \quad h = \frac{2T\cos\theta}{r\rho g} \qquad (1.24)$$

Since the angle of contact, θ is acute, $\cos\theta$ is positive and hence from the above formula, the value of h will be positive, *i.e.*, the liquid will rise up in the capillary tube. Also, it is seen that as r decreases, *i.e.* as the diameter of the tube decreases, the greater is the height to which the liquid will rise in the capillary tube. For example, when a capillary tube of glass is dipped in water, the level of water inside the capillary tube will rise.

If the angle of contact is obtuse, *i.e.* $\theta > 90°$, as in the case of liquids which do not wet the walls of the tube, $\cos\theta$ is negative, and hence from the above eqn.(1.23), the value of h will be negative. This shows that level of liquid will fall inside the capillary tube, *i.e.* there will be capillary depression. For example, when a capillary tube of glass is dipped in mercury, the level of mercury inside the capillary tube will be depressed.

In case of pure water and glass, the angle of contact θ is practically zero, so that from eqn.(1.23),

$$h = \frac{2T}{r\rho g} \quad or \quad T = \frac{1}{2}r\rho gh$$

This formula may be used to measure the surface tension of water by measuring its rise in a capillary tube.

Aliter: Let a glass capillary tube of uniform bore is dipped vertically in a liquid which wets the glass. Such a liquid rises up the tube forming a concave meniscus as already discussed. Let T be the surface tension of the liquid and EGF be the liquid meniscus in the tube in the final position. Since the force of surface tension tends to make the area of the surface-film minimum, it acts downwards at all the points of contact such as E and F [Fig.1.18 (a)], making an angle θ with the side of the tube. If r is the internal radius of the capillary tube, the liquid film touches the surface of the tube round a length $2\pi r$, the circumference of the circle of contact. Thus, the surface tension force T acts at every point of the circle of contact. According to Newton's third law, the reaction R acts on the liquid meniscus in the outward direction, at an angle θ to the vertical.

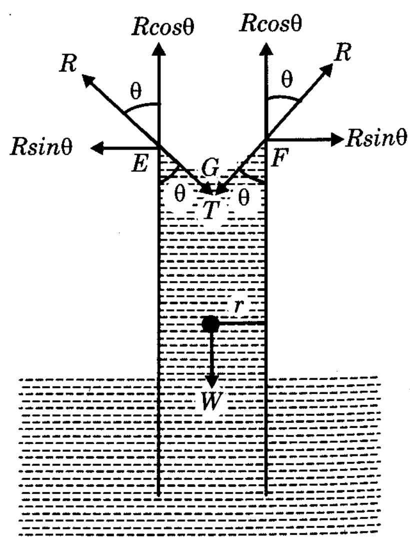


Fig: 1.18

Resolving R (= T) into two components:

(*i*) $R\cos\theta$ which acts at every point of the meniscus in the upward direction and is responsible for the rise of liquid in the capillary tube.

(*ii*) $R\sin\theta$ which acts at right angle to the length of the capillary tube in the outward direction.

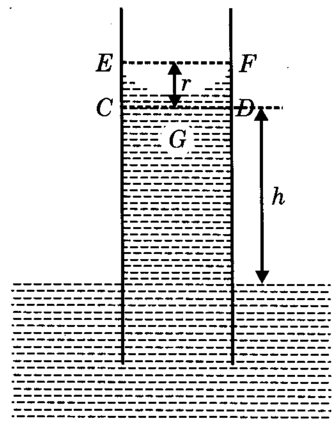


Fig: 1.18 (b)

Considering the entire length of the circle of contact at the meniscus, the horizontal components acting in one half of the circumference are equal and opposite to those acting in the other half and hence cancel each other. The vertical components in the upward direction are added up and thus the total vertical force acting along the circumference of the circle of contact is given by

$$F = R\cos\theta \times 2\pi r = T\cos\theta \times 2\pi r$$

This force supports the weight of the liquid column raised above the level outside the tube.

If h is the height of the liquid level in the tube from the horizontal surface in the vessel to the tangent plane at the bottom G of the meniscus EGF [Fig.1.18 (b)] and ρ the density of the liquid, then the volume of the liquid between the liquid surface in the vessel and the tangent plane at G = $\pi r^2 h$.

If the tube is narrow, the radius of curvature of the meniscus is approximately equal to the radius of the capillary tube i.e. r. Hence the volume of liquid raised in the capillary tube is

= Volume of cylinder of length h and radius r + (Volume of cylinder EDCF of length r and radius r – Volume of hemisphere of radius r)

$$= \pi r^2 h + \left(\pi r^2 r - \frac{2}{3}\pi r^3\right) = \left(\pi r^2 h + \frac{1}{3}\pi r^3\right)$$

$$= \pi r^2 \left(h + \frac{r}{3}\right)$$

∴ weight eight of the liquid raised in the capillary tube is W = Volume × density × g

$$= \pi r^2 \left(h + \frac{r}{3}\right)\rho g$$

In equilibrium, Total upward force = weight of liquid raised in the tube

$$\text{or} \qquad T\cos\theta \times 2\pi r = \pi r^2 \left(h + \frac{r}{3}\right)\rho g$$

$$\text{or} \qquad h + \frac{r}{3} = \frac{2T\cos\theta}{r\rho g}$$

$$\text{or} \qquad h = \frac{2T\cos\theta}{r\rho g} - \frac{r}{3}$$

This relation is known as ascent formula.

For narrow tubes, $r/3$ can be neglected in comparison to h.

$$\text{Hence} \qquad h = \frac{2T\cos\theta}{r\rho g}$$

This is the same as eqn. (1.24)

1.23. RISE OF LIQUID IN A TUBE OF INSUFFICIENT LENGTH

From the above, it is seen that when a capillary tube is dipped in a liquid, the liquid rises up the capillary tube until the weight of the liquid in the tube is just balanced by the by the force due to its surface tension. If θ be the angle of contact between the liquid and the tube, and R, the radius of curvature of the liquid meniscus in the tube, we have $r=R\cos\theta$ (eqn. 1.23, Art.1.21), where r is the radius of the tube, and from eqn. 1.24 (Art. 1.21) the rise of liquid in the capillary tube is given by,

$$2T/R = h\rho g$$

Where T = surface tension of the liquid, h= height to which the liquid rises in the tube, and ρ = density of the liquid.

$$\text{Hence, clearly } Rh = \frac{2T}{\rho g} = \text{a constant}$$

When the length of the capillary tube is greater than h, R remains constant and the liquid rises to a height h, so as to satisfy the above relation. But, if the length of the tube be smaller than h, say h', the only variable in the above relation is R, because now $h = h'$, the length of the tube (a constant) and so is θ a constant for the given liquid and the tube. The liquid rises up to the top of the tube and just spreads over the walls of the tube at the top and its meniscus acquires a new radius of curvature R', such that

$$R'h' = Rh = \frac{2T}{\rho g} = \text{a constant}$$

Thus, when $h' <$ h, $R' >$ R *i.e.*, The radius of curvature increases and the liquid will not overflow.

1.24. ENERGY REQUIRED TO RAISE A LIQUID IN A CAPILLIARY TUBE

When a capillary tube is dipped vertically in a liquid which wet the walls of the capillary tube, the liquid level rises inside the capillary tube. This rise in the level, obviously, takes place against gravity and, therefore, the liquid must gain potential energy as it rises in the capillary tube. According to the law of conservation of energy, energy can only be converted from one form to another and it cannot be created. The question therefore arises, where from did the liquid get this increase in its potential energy? The explanation to this is as follows:

At the meniscus of the liquid, there are three surfaces of separation to be considered, *viz.*

(*i*) An air-liquid surface (*ii*) An air-glass surface and (*iii*) A glass-liquid surface, each having its own surface tension, different from the others, and equal to its free surface energy per unit area.

As the liquid rises in the capillary tube, the area of liquid-glass surface increases whereas the area of air-glass surface decreases by the same amount. Moreover as the plane liquid surface in the tube acquires a curvature, *i.e.*, it becomes concave, the air-liquid surface also increases. Therefore, the surface energy of glass-liquid surface and liquid-air surface increases whereas the surface energy of glass-air surface decreases by the same amount. Thus the energy required to raise the liquid in the capillary tube is obtained from the decrease in the surface energy of glass-air surface. In other words, the lost energy appears in the form of gravitational potential energy of the raised liquid column in the capillary tube.

When a liquid falls down in a capillary tube, air-glass surface increases and liquid-glass surface decreases by the same amount and thus there is a net increase in the surface energy of the whole system. This energy is derived from the depression of the liquid inside the tube, whose gravitational energy is thus decreased by the same amount.

1.25. PRACTICAL APPLICATIONS OF CAPILLARITY

Some of the practical applications of capillarity are as follows:

(*i*) The oil in a lamp or stove rises in the wick to its top by rising in the long narrow spaces between the threads of the wick.

(*ii*) The fine pores of a blotting paper act like capillary tubes and hence ink is absorbed by the blotting paper by capillary action.

(*iii*) A towel soaks water on account of capillary action due to the fine pores between the threads of a towel.

(*iv*) Sap and water rises to the top of the leaves of the tree by capillary action.

(*v*) Ploughing of fields is essential for preserving moisture in the soil. By ploughing, the fine capillaries of the soil are broken and hence water from within the soil shall not rise and evaporate off.

(*vi*) Sand is drier soil than clay. This is because the particles of sand are not very fine as in the case of clay, so as to draw water by capillary action.

1.26. FACTORS AFFECTING SURFACE TENSION

The various factors affecting the surface tension of a liquid are:

1. The presence of impurities either on the surface or dissolved in it, considerably affect the force of surface tension of a liquid, depending upon the degree of contamination.

When a substance is dissolved in a liquid, it may increase or decrease the surface tension depending on the nature of the dissolved substance. Many organic compounds when dissolved in water reduce the tension, while most inorganic salts, however, increase the surface tension of the liquid in which they dissolve. For example, when sodium chloride is dissolved in water, it increases the surface tension of water. But when phenol is dissolved in water, it decreases the surface tension of water.

The contamination of the surface also has a marked effect on the surface tension of a liquid. Contaminants such as oil, grease etc. easily spread over liquids and reduce the surface tension to a great extent.

2. The surface tension of a liquid also depends on the state of electrification of the liquid. Electrification of a liquid is found to cause an outward force on the liquid, thus trying to increase its surface area. This force thus acts opposite to the force of surface tension of the liquid and, consequently the surface tension of the liquid is reduced.
3. The surface tension of all liquids is found to decrease linearly with rise of temperature. For small temperature ranges, the variation in surface tension with temperature is given by,

$$T_\theta = T_0(1 - \alpha t)$$

Where T_0 and T_θ are the surface tensions of the liquid at $0°C$ and $\theta\,°C$ respectively and α is the temperature coefficient of surface tension.

At a certain temperature, called *critical temperature* (θ_c), the tension vanishes and no surface phenomenon is observed. A relation representing the variation of surface tension with temperature in terms of the critical temperature was given by Ferguson as,

$$T_\theta = A\left(1 - \frac{\theta}{\theta_c}\right)^n$$

Where A is a constant and n is a constant for a given liquid. The value of n is different for different liquids having an average value of 1.21.

A more accurate relation was given by Eötvos and improved by W. Ramsay and J. Shields as follows,

$$T\left(\frac{M}{P}\right)^{\frac{3}{2}} = K(\theta_c - \theta - \delta)$$

Where M is the molecular weight and ρ is the density, of the liquid; K is a universal constant of approximate value 2.2 and δ is a constant, whose value lies between 6 and 8 for most liquids. The relation shows that surface tension of the liquid vanishes at $\theta = \theta_c - \delta$, *i.e.*, at a temperature little below the critical temperature.

1.27 SOLVED EXAMPLES

1. The surface What height of water column produces the same pressure as a 76cm column of mercury? (density of mercury = 13.6×10^3 kgm^{-3})

Solution: Here, $P = h\,r\,g = \frac{76}{100} m \times 13.6\ 13.6 \times 10^3 \text{ kgm}^{-3} \times 9.8\text{ms}^{-2}$

$$= 101292.8 \text{ Nm}^{-2}$$

Let h' = height of water column; density of water $\rho' = 10^3$ kgm^{-3}

$$= 101292$$

$$P = h'\,\rho'\,g \quad \Rightarrow h' = P/\rho' g$$

or, $$h' = \frac{101292.8}{1000 \times 9.8} = 10.336\text{m}$$

Hence, 10.336*m* of water column produces the same pressure as a 76cm high column of mercury.

2. Calculate the pressure exerted by water on a fish 10 m below the surface of a pond. Take atmospheric pressure = 1.013×10^5 Pa.

Solution: Let P be the pressure on the fish at a depth of 10m.

$$\therefore \quad P = P_a + h\,\rho\,g$$

$$= 1.013 \times 10^5 + 10 \times 1000 \times 9.8$$

$$= 1.993 \times 10^5 \text{ Pa}$$

3. Two liquids of specific gravity 1.2 and 0.84 are poured into the limbs of a U-tube until the difference in levels of their upper surfaces is 9 cm. What will be the heights of their respective surfaces above the common surface in the U-tube? What is the pressure aat the common surface? (take g = 10ms^{-2})

Solution: Let h_1 and h_2 be the heights of the denser and lighter liquid respectively above the common level. Then,

$$= 1.01$$

$$h_2 - h_1 = 9.0 \qquad .....(i)$$

At the common level surface, $h_1\rho_1 g = h_2\rho_2 g$

i.e., $h_1 \times 1.2 \times g = h_2 \times 0.84 \times g$

or, $h_1 = (0.84/1.2)\,h_2$

From (*i*), $h_2 - 0.7h_2 = 9$ or, $0.3\,h_2 = 9$

or, $h_2 = 30$cm and $h_1 = 0.7 \times 30 = 21$cm

Pressure at the common surface = $h_{22}g = 0.30 \times (0.84 \times 10^3) \times 10$

$$= 2520 \text{ Nm}^{-2}$$

4. A hydraulic lift is designed to lift cars with a maximum mass of 4×10^3 kg. The area of the piston carrying the load is 5×10^{-2}m^2. How much pressure the smaller piston will bear?

Solution: here, maximum mass, m = 4×10^3 kg

$$h_2 - h_1 = 9.0$$

$\therefore$ Maximum force = mg = $4\times10^3 \times 9.8$

$$= 39.2 \times 10^3$$

And, Area = $5\times10^{-2}\,m^2$

$\therefore$ Maximum force = $\dfrac{\text{Maximum force}}{Area}$

$$= \frac{39.2\times10^3}{5\times10^{-2}} = 7.84 \times 10^5\ Nm^{-2}$$

5. The density of ice is $917kgm^{-3}$. What fraction of the volume of a piece of ice will above water, when floating in fresh water?

Solution: Let ρ be the density of ice = 917 kgm-3, and ρ' be the density of water = 1000 kg^{-3}.

Also, let V be the total volume of ice and v be the volume of ice above the water. Then, volume of the water displaced by the immersed portion of the ice = (V – v).

According to the law of floatation, weight of ice = weight of the water displaced

$\therefore$ $V \times 917\ g = (V - v) \times 1000 \times g$

or, $1000v = 1000V - 917V = 83V$

or, $\dfrac{v}{V} = \dfrac{83}{1000} = 0.083$

Here, about 8.3% of the ice will float above the surface of water.

6. A piece of pure gold of density, $19.3gcm^{-3}$ is suspected to be hollow inside. It weighs 38.250 g in air and 33.865 g in water. Calculate the volume of the hollow portion of the gold, if any.

Solution: Here, density of pure gold, $\rho = 19.3gm^{-3}$

Mass of the gold piece (in air), M = 38.250g

$\therefore$ Volume of gold piece, $V = \dfrac{M}{\rho} = \dfrac{38.25}{19.3} = 1.982cm^3$

Therefore, the apparent loss in weight of the gold piece in water,

$$= 38.250 - 33.865$$
$$= 4.385$$

As density of water is $1gcm^{-3}$, therefore, the volume of displaced water

$$= \frac{4.385}{1} = 4.385cm^3$$

$\therefore$ Volume of the hollow portion of the gold = 4.385 – 1.982 = $2.403cm^3$

7. Calculate the work done in blowing a soap bubble from radius 2cm to 5om. Surface tension of soap solution = $6.0 \times 10^{-2} Nm^{-2}$.

Solution: Initial radius of bubble, r_1 = 2cm = 2×10^{-2}m

Final radius of bubble, r_2 = 5cm = 5×10^{-2}m

Since a soap bubble has two free surfaces, therefore, increase in surface area of bubble

$$= 2\left(4\pi r_2^2 - 4\pi r_1^2\right) = 2 \times 4\ p\ \left(r_2^2 - r_1^2\right)$$

$$= 2 \times 4 \times \frac{22}{7}\ (22 \times 10^{-4} - 4 \times 10^{-4})$$

$$= 528 \times 10^{-4} m^2$$

$\therefore$ Work done = surface tension × increase in surface area

$$= 6.0 \times 10^{-2} \times 528 \times 10^{-4} = 3.168 \times 10^{-3} J$$

8. Caluculate the energy evolved when 8 droplets of water, each of radius 0.5mm coalesce to form one spherical drop. Surface tension of water = $72 \times 10^{-3} Nm^{-1}$.

Solution: Here, Surface tension, $T = 72 \times 10^{-3} Nm^{-1}$, r = 0.5mm = 0.5×10^{-3}m

Let R be the radius of the big drop formed.

Now, volume of 8 small drops = volume of the big drop

or, $$\frac{4}{3}\pi R^3 = 8 \times \frac{4}{3}\pi r^3$$

or, $$R = 2r = 2 \times 0.5 \times 10^{-3} = 10^{-3} m$$

Decrease in surface area $= 8 \times 4p r^2 - 4pR^2 = 4p\ [8\ r^2 - R^2]$

$$= 4 \times 3.14 \times [8(0.5 \times 10^{-3})^2 - (10^{-3})^2]$$

$$= 12.56 \times 10^{-6} m^2$$

$\therefore$ Energy evolved = $T \times$ decrease in surface area

$$= 72 \times 10^{-3} \times 12.56 \times 10^{-6}$$

$$= 9.04 \times 10^{-7} J$$

9. Caluculate If the excess pressure inside a spherical soap bubble of radius 1cm is balanced by . that due to a column of oil of specific gravity 0.9, 2mm high, calculate the surface tension of water.

Solution: Here, r = 1cm = 10^{-2}m; density of oil, = $0.9 \times 10^3 kgm^{-3}$; h = 2mm = 2×10^{-3}m

Pressure due 2mm column of oil, $P = h\rho g = 2 \times 10^{-3} \times 0.9 \times 10^3 \times 9.8$

$$= 17.64 Pa$$

Let T be the surface tension of soap solution.

$\therefore$ Energy evolved, $$p = \frac{4T}{r}$$

or, $$T = \frac{pr}{4} = \frac{17.67 \times 10^{-2}}{4}$$

$$= 4.42 \times 10^{-2} Nm^{-1}$$

10. A small hollow sphere which has a small hole in it is immersed in water to a depth of 40cm before any water is penetrated into it. If the surface tension of water is $0.073 Nm^{-1}$, find the radius of the hole.

Solution: Here, $T = 0.073\ Nm^{-1}$; $h = 40cm = 40\times10^{-2}m$; $\rho = 10^3 kgm^{-3}$

Pressure exerted by 40cm column of water $= h\,\rho\,g$

$$= 40\times10^{-2}\times10^3\times9.8$$

$$= 3920\ Pa$$

Excess pressure inside the air bubble $= \dfrac{2T}{r} = \dfrac{2\times0.073}{r} = \dfrac{0.146}{r}$

Now, $$3920 = \frac{0.146}{r}$$

or, $$r = \frac{0.146}{3920} = 3.7\times10^{-5}m$$

11. A capillary tube of inner diameter 0.5mm is dipped in a liquid of specific gravity 13.6, surface tension $545 dynecm^{-1}$ and angle of contact 130°. Find the depression or elevation in the tube.

Solution: Here, $r = \dfrac{d}{2} = \dfrac{0.5}{2} = 0.25mm = 0.025cm$; T = 545 $545 dynecm^{-1}$, $\rho = 13.6 gcm^{-3}$

$$\theta = 130^\circ$$

$\therefore$ $$cos130^\circ = -0.6428$$

Now, $$T = \frac{r(h + r/3)\rho g}{2\cos\theta}$$

$\therefore$ $$h = \frac{2T\cos\theta}{r\rho g} - \frac{r}{3}$$

$$= \frac{2\times545\times(-0.6428)}{0.025\times13.6\times980} - \frac{0.025}{3}$$

$$= -2.111cm$$

Hence, the liquid will get depressed in the tube.

1.28 SAMPLE QUESTIONS

1. What is a fluid? Define thrust and pressure at a point. Why the thrust is always perpendicular to the surface in contact with it?
2. What is pressure? Give its units and dimensions. Deduce an expression for the hydrostatic pressure at a depth h below the liquid surface. Show that the pressure difference between two points of a liquid varies as the vertical distance between them.
3. State and explain Archimedes' principle. Also state the law of floatation.
4. Show that the pressure of a liquid is proportional to the height of the liquid column. Also discuss the effect of gravity on the pressure of a liquid.
5. State and prove Pascal's law. Give two applications of the law.
6. Define surface tension and surface energy. Show that surface tension is numerically equal to surface energy.
7. What is surface tension of a liquid? State its units. Explain surface tension on the basis of the molecular theory.
8. Show that the excess pressure acting on the curved surface of a curved membrane is given by, $p = 2T\left(\frac{1}{r_1} \pm \frac{1}{r_2}\right)$ where r_1 and r_2 are the radii of curvature and T, the surface tension of the membrane.
9. Calculate the excess pressure acting on the curved surface of (i) a liquid drop, (ii) an air bubble inside a liquid and, (iii) a soap bubble.
10. What is angle of contact? Why the surface of water is concave and surface of mercury is convex when it is kept in contact with solid?
11. What is capillarity? Derive an expression for the rise of a liquid in a capillary tube of uniform diameter. Explain from where the energy comes when a liquid rises against gravity in a capillary tube?
12. What do you mean by excess pressure inside a liquid drop? Describe Jaeger's experiment for the determination of surface tension.
13. Explain why:
    (a) small liquid drops are spherical in shape, but big drops are flat?
    (b) a needle when placed on blotting paper and then gently placed on the surface of water floats?
    (c) small insects are able to walk on the water surface?
    (d) does the hairs of a shaving brush cling together when taken out of water?
    (e) mercury does not wet glass but water does?
    (f) fields are ploughed?
    (g) pressure difference exists between the two sides of a curved surface?
    (h) sand is a drier soil than clay?

1.29 SAMPLE PROBLEMS

1. A cylindrical jar of cross-sectional area $50cm^2$ is filled with water to a height of 20cm. It carries a piston of negligible mass. Neglecting atmospheric pressure, calculate the pressure at the bottom of the jar when a mass of 1kg is placed on the piston. (**Ans.** $2.94\times10^3Nm^{-2}$)

2. A column of water 40cm high supports a 30cm column of an unknown liquid. What is the density of the liquid? (**Ans.** $1.33\times10^3kgm^{-3}$)

3. Calculate the pressure on a diver 100m below the surface of the ocean. (**Ans.** $11\times10^5Nm^{-2}$)

4. A base of rectangular vessel measures 10cm×18cm. Water is poured into it upto a depth of 4cm. What is the pressure and thrust on the base of the vessel? (Take $g=10ms^{-2}$)

   (**Ans.** 400 Pa & 7.2N)

5. During blood transfusion, the bottle of the blood is hanged at a certain height with a stand. If the guage pressure at a point where the needle is inserted in the vein of a patient is 1500Pa, find the height of the bottle of the blood so that the blood may enter into the vein of. (Density of blood= $1060kgm^{-3}$). (**Ans.** 14.4cm)

6. A hydraulic lift is designed to lift cars with a maximum mass of 4000kg. The area of cross-section of the piston carrying the load is $5\times10^{-2}m^2$. How much pressure the smaller piston will bear? (**Ans.** 7.84×10^5Pa

7. The average mass that must be lifted by a hydraulic lift is 80kg. If the radius of the larger piston is five times that of the smaller piston, what is the minimum force that must be applied? (**Ans.** 31.4N)

8. A solid floats in water with ¾ of its volume below the surface of water. Calculate the density of the solid. (**Ans.** 750 kgm^{-3})

9. A spring balance reads 10kg when a bucket of water is suspended from it. What is the reading of the balance when,—*(i)* an ice cube of mass 1.5kg is put into the bucket. *(ii)* an iron piece of mass 7.8kg suspended by another string is immersed with half its volume inside the water in the bucket? (relative density of iron = 7.8) (**Ans.** 11.5kg; 10.5kg)

10. A cylinder of length 10cm is immersed in mercury. It is found that the cylinder floats with 4.3cm of its length remaining above the surface of mercury. Find the density of the material of the cylinder. (density of mercury = 13.6 g/cc) (**Ans.** 7.75g/cc)

11. What amount of energy will be liberated if 1000 droplets of water, each $10^{-8}m$ in diameter coalesce to form one spherical drop? (surface tension of water = $72\times10^{-3}Nm^{-1}$)

    (**Ans.** $2.035\times10^{-14}J$)

12. There is a minute circular hole at the bottom of a small hollow vessel. The vessel has to be immersed in water to a depth of 0.4m so that no water penetrates inside. Calculate the radius of the hole. (surface tension of water = $72\times10^{-3}Nm^{-1}$). (**Ans.** $1.37\times10^{-5}m$)

13. A soap bubble (surface tension = 25dyne/cm) is slowly enlarged from a radius of 3cm to a radius of 5cm. Calculate the work done in the process. (**Ans.** 10.05×10^3erg)

14. A soap bubble of surface tension $0.026Nm^{-1}$ is slowly enlarged from a radius of 0.01m to a radius of 0.10m. Calculate the work done in the operation. Explain why this is less if the operation is done slowly? (**Ans.** $6.36\times10^{-3}J$)

15. A glass plate of length 10cm, breadth 1.54cm and thickness 0.20cm weighs 8.2g in air. If it is held vertically, with its long side horizontal and its lower half immersed in water, what will be its apparent weight? (surface tension of water = $72\times10^{-3}Nm^{-1}$) (**Ans.** 8.18 gwt)

16. A wire ring of radius 3cm is rested flat on the surface of a liquid and is then raised. The pull required 3.03g more before the film breaks than it is after. Calculate the surface tension of the liquid. (**Ans.** 78.8 dyne/cm)

17. Two spherical soap bubbles of diameter 10cm and 6cm respectively are formed, one at each end of a narrow horizontal glass tube. What is the pressure difference between the ends of the tube? (surface tension = 30 dyne/cm) (**Ans.** 16 dyne/cm^2)

18. (a) Calculate the excess of pressure in a soap bubble of 2mm radius, if the value of T is $0.03Nm^{-1}$. (**Ans.** 60N/m^2)

    (b)Calculate the excess pressure inside a soap bubble of radius 3cm. The surface tension of the soap solution is 0.02 Nm^{-1}. Find also the surface energy of the soap bubble. (**Ans.** 2.67N/m^2; 4.52×10^{-4}J)

19. The pressure of air inside a soap bubble of diameter 0.7cm is 8mm of water above the atmospheric pressure. Determine the surface tension of soap solution. (**Ans.** 0.069 Nm^{-1})

20. A soap bubble is spherical in shape and has a diameter of 10cm. If the surface tension of the surface separating soap solution and air is 0.04 SI units, what is the excess pressure of the air in the bubble over the atmospheric pressure? (**Ans.** 3.2 N/m^2)

21. Find the pressure in a spherical bubble of of 0.002 cm radius at a depth of 85cm in an oil of density $800kgm^{-3}$ and surface tension 0.025 Nm^{-1}. (**Ans.** 9164 N/m^2)

22. What is the difference of pressure between the inside and outside of of a spherical drop of water of radius 1mm? (surface tension of water = $72\times10^{-3}Nm^{-1}$) (**Ans.** 146 N/m^2)

23. A capillary tube of uniform bore of diameter 0.5mm stands vertically in a wide vessel containing a liquid of surface tension $0.03Nm^{-1}$. The liquid wets the tube and has a specific gravity of 0.8. Calculate the rise of liquid in the tube. (**Ans.** 3.061cm)

24. What should be the radius of a capillary tube so that water will rise to a height of 8cm in it? (**Ans.** 0.018 cm)

25. A capillary tube is dipped in water. Water rises to a height of 4 cm above the surrounding liquid. If the angle of contact is zero and the radius of the tube is 0.1 mm, what is the surface tension of the liquid? (**Ans.** 0.0196 N/m)

26. In Jaeger's experiment, a capillary tube of internal diameter 5×10^{-2}cm dips 3cm inside water contained in a beaker. The difference in level of manometer, when the bubble is released, is 0.09m. Calculate the value of surface tension. (**Ans.** 73.5×10^{-3} N/m)

Chapter 2

Hydrodynamics— Viscocity

2.1. STREAMLINE AND TRUBULENT FLOW

When a liquid flows such that the velocity of every particle passing through a point in the liquid is constant, both in magnitude and direction, then the flow of the liquid is called steady or streamline flow.

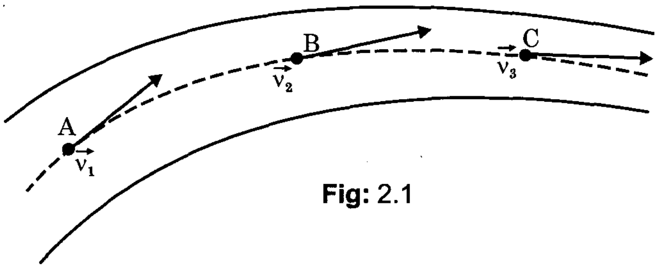


Fig: 2.1

In such a case, the path followed by the particles of the liquid represents the direction of flow of the liquid at any particular point in the liquid. This path followed by the particles of a liquid (or a fluid, in general), in a streamline flow is called a streamline. More correctly, a streamline is a curve, the tangent to which at any point and at any instant, gives the direction of flow of the liquid at that point. A streamline, therefore, may be straight or curved according as the lateral pressure on it is the same or different.

Consider a liquid flowing through a tube as shown in Fig.2.1. Let ABC represent a streamline in the liquid *i.e.*, a path along which the particles of the liquid would move. Let $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ be the velocities of a particle at A, B and C respectively. If all the preceding or succeeding particles of the liquid move along A, B and C with the same velocities, then the flow of the liquid is known as steady or streamline flow. In a streamline flow, the energy needed to drive the liquid is used up only in overcoming the viscous drag between the layers.

Characteristics of a streamline: (i) A tangent drawn at any point on the streamline gives the direction of the velocity of the liquid particles at that point.

(ii) No two streamlines can cross each other. If they intersect each other, then at the point of intersection, two tangents may be drawn, which represent two directions of liquid flow at that point, which is not possible.

A bundle of streamlines having the same velocity of the liquid particles over any cross-section perpendicular to the direction of flow is called a *tube of flow*. The flow of the liquid remains streamline so long as the velocity of flow of the liquid does not exceed a certain value, called its *critical velocity*. The value of critical velocity is different for different liquids. If a liquid is flowing over a horizontal surface with a steady flow such that different layers of the liquid move with different velocities, but these layers do not mix with each other, then the flow of the liquid is called a *laminar flow*. In this type of flow, the velocity of flow is always less than the critical velocity of the liquid and one layer simply slides over the other layer of the liquid.

When a liquid moves with a velocity greater than its critical velocity, the motion of the particles of liquid becomes disorderly or irregular. Such a flow is called turbulent flow.

In a turbulent flow, the path and velocity of the particles of the liquid change continuously and haphazardly with time from point to point. The flow becomes zigzag and sinuous in character; most of the energy maintaining the flow is dissipated in the formation of eddies and whirlpools in the liquid and, only a small fraction of the energy is available for driving the liquid forward.

2.2. RATE OF A FLOW OF A LIQUID

The *rate of flow* of a liquid is defined as *the volume of the liquid that flows across any section of a pipe in unit time.* Rate of flow is actually the volume rate of flow of the liquid, i.e. the volume of the liquid flowing per second across any section of the pipe. It is usually represented by the letter Q or V.

Considering the liquid to be incompressible, if its velocity of flow be v, in a direction perpendicular to two sections A and B, (Fig.2.2.) of area a, and distance l apart, and if t be the time taken by the liquid to flow from A to B, we have $vt = l$

Obviously, the volume of liquid flowing through the section AB, in *this* time, is equal to the cylindrical column $AB = l \times a = vt \times a$. This, therefore, is the volume of liquid flowing across the section in time t.

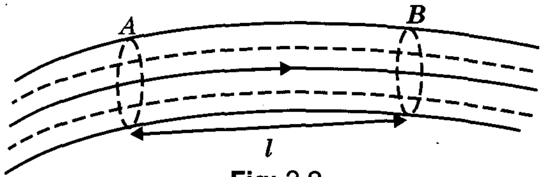


Fig: 2.2

∴ *Volume rate of flow of liquid,*

$$Q \text{ or } V = V = \frac{vt \times a}{t} = v \times a$$

= *velocity of liquid × area of cross-section of the pipe.*

Sometimes, the rate of flow of a liquid is also expressed in terms of the mass of the *liquid flowing across any section in unit time* and is referred to as its *mass rate of flow*. Thus,

mass rate of flow of a liquid = mass of liquid flowing across any section per unit time

= volume rate of flow of liquid × density of liquid

= velocity of liquid × area of across-section × density of liquid = $v \times a \times \rho$.

2.3. EQUATION OF CONTINUITY OF FLOW

This is a fundamental equation of flow in hydrodynamics and it expresses the general physical law of conservation of matter/mass. For an *incompressible fluid i.e., a liquid*, it may be deduced as follows:

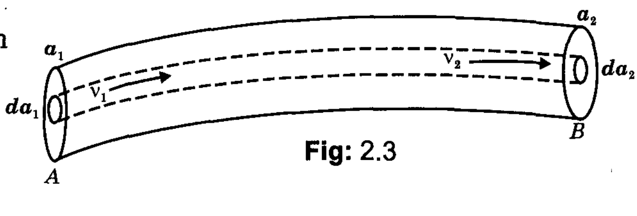


Fig: 2.3

Consider a liquid to be flowing through a pipe AB, (Fig.2.3.) of varying areas of cross-section. Let a_1 and a_2 be its areas of cross-section at sections A and B and consider an infinitesimally small tube of flow (shown dotted) of cross-sectional areas da_1 and da_2 at its two ends. Then, if the liquid covers distance ds_1 and ds_2 in time dt, at the two ends with velocities of flow v_1 and v_2 at sections A and B respectively, and if ρ_1 and ρ_2 be the densities of the liquid at A and B, we have

Mass of liquid entering flow-tube at end A per unit time = $da_1 . ds_1 . \rho_1 . / dt$

$$= da_1 . v_1 . \rho_1 \qquad [\because ds_1 / dt = v_1]$$

and *mass of liquid leaving flow-tube at end B per unit time* = $da_2 . ds_2 \rho_2 / dt$

$$= da_2 . v_2 . \rho_2 \qquad [\because ds_2 / dt = v_2]$$

∴ *mass of liquid entering the whole section A per sec, i.e. mass rate of flow at A*

$$= \int_0^{a_1} da_1 . v_1 . \rho_1 = a_1 . v_1 . \rho_1$$

and, *mass of liquid leaving the whole section B per sec, i.e., mass rate of flow at B*

$$= \int_0^{a_2} da_2 . v_2 . \rho_2 = a_2 . v_2 . \rho_2$$

Since a liquid is incompressible, therefore, $\rho_1 = \rho_2$ and since there is no source or sink in between the sections A and B, we have, from the law of conservation of mass,

$$a_1 v_1 = a_2 v_2 = V \text{ (or } Q)$$

i.e., the *rate of flow of the liquid at A = rate of flow of the liquid at B.*

This is called the *equation of continuity* and it states that, for an incompressible fluid i.e. a liquid, *the quantity of liquid entering one end of the pipe per second is the same as that leaving the pipe at the other end per second.*

Obviously, what is true of sections A and B is also true of all other sections of the pipe too. It follows, therefore, that *the rate of flow of an incompressible and mobile fluid is the same throughout a pipe in the case of steady or streamline flow.*

Further, it follows straight away from the above that, $\frac{v_1}{v_2} = \frac{a_2}{a_1}$ *i.e., the velocity of the liquid varies inversely as the cross-section of the pipe.* Thus as the area of cross-section of the pipe becomes larger, the speed of the liquid becomes smaller and vice-versa.

Examples: (*i*) Velocity of a liquid is greater in the narrow section of a tube as compared to the velocity of the liquid in the broader section of the tube.

(*ii*) Deep waters run slow because the area of cross-section increases where water is deep and hence the velocity or speed decreases.

In the case of a gas, since the density changes with pressure (due to its high compressibility), it is not the volume but the mass of the gas that remains constant through any section of the pipe. So that, if ρ_1 and ρ_2 be the densities of the gas at the two sections A and B respectively in the figure above, we have $a_1 v_1 \rho_1 = a_2 v_2 \rho_2$ or $V_1 \rho_1 = V_2 \rho_2$.

2.4. ENERGY POSSESSED BY A LIQUID

A liquid in motion has three types of energy, *viz.*, (*i*) Kinetic Energy, (*ii*) Potential Energy and (*iii*) Pressure Energy.

(*i*) **Kinetic Energy**. It is the energy possessed by a liquid by virtue of its motion or velocity. Since a liquid has inertia, therefore, it possesses kinetic energy when in motion.

If a liquid of mass m and density ρ is flowing with a velocity v, then the kinetic energy of the liquid will be $\frac{1}{2}mv^2$. Hence, kinetic energy per unit mass = $\frac{1}{2}v^2$ and kinetic energy per unit volume = $\frac{1}{2}\rho v^2$

(*ii*) **Potential Energy**. It is the energy possessed by a liquid by virtue of its height or position above the surface of earth or any reference level taken as zero level.

If m is the mass of a liquid at a mean height h above some reference horizontal level and V is its volume, then the potential energy of the liquid is mgh. Hence, its potential energy per unit mass = gh, and its potential energy per unit volume $= \rho g h$.

(*iii*) **Pressure Energy**. It is the energy possessed by a liquid by virtue of its pressure or hydrostatic pressure.

Consider an incompressible liquid of density ρ which is contained in a vessel provided with a side tube S and fitted with a frictionless piston of area of cross-section a (Fig.2.4.). Let h be the height of free surface of the liquid in the vessel above the axis of the tube. Then, the hydrostatic pressure on the piston $p = h \rho g$.

Let the liquid be pushed inside the vessel by slowly pushing the piston inward through a small distance dx. Since the process is slow, hence the kinetic energy acquired by the liquid in the process is negligible.

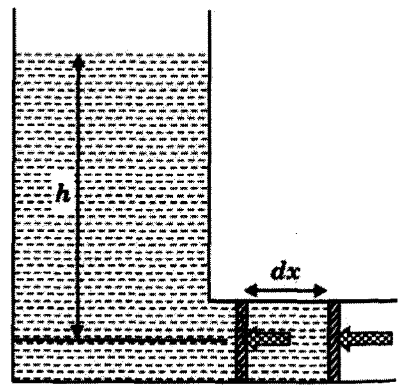


Fig. 2.4

$$\text{Now, the volume of liquid pushed into the tank } = a \times dx$$

$$\text{and, mass of liquid pushed into the tank } = a \times dx \times \rho$$

$$\text{Force on the piston, } F = p \times a$$

$$\therefore \text{ Work done in pushing the liquid into the tank } = F \times dx = p \times a \times dx$$

This work done in pushing the liquid against the pressure p into the tank without imparting it any kinetic energy (K.E.) becomes the pressure energy of mass $a\,dx\,\rho$ and of volume $a\,dx$, of the liquid.

Hence, the pressure per unit mass of the liquid $= \frac{p}{\rho}$, and pressure per unit volume = p.

Thus, the pressure energy per unit volume of the liquid is equal to the hydrostatic pressure due to the liquid.

2.5. BERNOULLI'S THEOREM

This theorem states that for the streamline flow of an ideal liquid (incompressible and non-viscous), the total energy (i.e., the sum of pressure energy, kinetic energy and potential energy) per unit mass of the liquid remains constant at every cross-section throughout the flow.

i.e., (pressure energy + kinetic energy + potential energy) per unit mass = constant

$$\text{or} \quad \frac{p}{\rho} + \frac{1}{2}v^2 + gh = \text{constant}$$

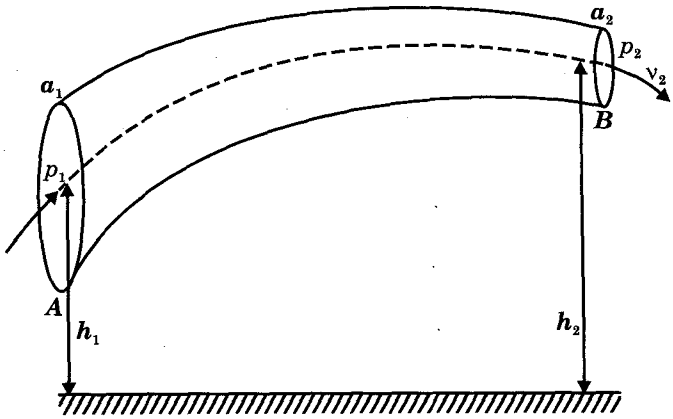


Fig. 2.5

Proof. Consider a tube *AB* of varying area of cross-section through which an ideal liquid is in streamline flow from *A* to *B* as shown in Fig.2.5. Let p_1, a_1, h_1, v_1 and p_2, a_2, h_2, v_2 be the pressure, area of cross-section, height and velocity of flow at points A and B respectively. Since the liquid is flowing from *A* to *B*, therefore, $p_1 > p_2$. According to the equation of continuity, the mass m of the liquid crossing per second through any section of the tube is given by,

$$m = a_1 v_1 = a_2 v_2 \rho$$

$$\text{or,} \quad a_1 v_1 = a_2 v_2 = \text{m}/\rho \qquad (2.1)$$

Force acting on the liquid at section A causing the liquid to move through the tube towards the section $B = p_1 a_1$. The mass *m* of the liquid entering the tube through section *A* moves forward and travel *a* distance v_1 per second parallel to the axis of the tube and along the direction of the force $p_1 a_1$.

$\therefore$ Work done per second on the liquid at section $A = p_1 a_1 \times v_1 = p_1 a_1 v_1$.

At the section *B*, the mass *m* of the liquid leaves the tube and moves forward a distance v_2 per second. But the flow is opposite to that of the force $p_2 a_2$ acting over the section. Thus, in this case,

The work done per second by the liquid at section $B = p_2 a_2 \times v_2 = p_2 a_2 v_2$

$\therefore$ The net work done per second by the pressure forces on the liquid in moving it from section A to section $B = p_1 a_1 v_1 - p_2 a_2 v_2 = (p_1 - p_2)\dfrac{m}{\rho}$ {using (2.1)}

Since $h_2 > h_1$, therefore there is a gain in the gravitational potential energy of mass m of the liquid as it moves from sections A to B. Hence the increase in potential energy per second of the liquid in moving from A to $B = mg\,(h_2 - h_1)$.

Also, as $A > B$, therefore $v_2 > v_1$ *i.e.*, there is an increase in the kinetic energy of mass m of the liquid in moving from A to B. Hence the increase in kinetic energy per second of the liquid in moving from A to $B = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$.

This gain in energy of the liquid in moving from A to B is obviously at the expense of the pressure energy of the liquid and thus represents a loss in the pressure energy of the liquid. According to the principle of conservation of energy,

total loss in energy per second = total gain in energy per second

or, $$(p_1 - p_2) = mg\,(h_2 - h_1). + \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

whence, $$hg + \frac{p_1}{\rho} + \frac{1}{2}v_1^2 = hg + \frac{p_2}{\rho} + \frac{1}{2}v_2^2$$

i.e., $$hg + \frac{p}{\rho} + \frac{1}{2}v^2 = \text{constant} \qquad (2.2)$$

Hence, the Bernoulli's equation is established.

From equation (2.2) it is seen that the sum of the three forms of energy of a perfectly mobile and incompressible liquid in motion is constant and thus, when one form of energy decreases, the other forms will increase and vice-versa. Hence the three forms of energy of a liquid are mutually convertible to each other.

Dividing eqn. (2.2) by g, we have

$$hg + \frac{p}{\rho g} + \frac{1}{2}\frac{v^2}{g} = \text{constant} \qquad (2.3)$$

Each of these quantities has the dimensions of length and is called a head — h is called gravitational head, $\dfrac{p}{\rho g}$ is called pressure head and $\dfrac{v^2}{2g}$ is called velocity head.

Thus, gravitational head + pressure head + velocity head = constant

Hence, Bernoulli's theorem can also be stated as, in the streamline flow an ideal liquid, the sum of the gravitational head, the pressure head and the velocity head at every cross-section is a constant.

If the liquid is flowing through a horizontal tube, the gravitational head is a constant, so that from eqn. (2.3), we have

$$\frac{p}{\rho g}+\frac{v^2}{2g}=\text{constant} \qquad \text{or,} \quad p+\frac{1}{2}\rho v^2=\text{constant}$$

This relation shows that greater pressure corresponds to smaller velocity and vice-versa. Thus, in the streamline flow of an ideal liquid through a horizontal tube, the velocity increases where pressure decreases and vice-versa. For example, when a liquid flows through a horizontal tube having a constriction, the velocity of the liquid in the constricted part increases and hence the pressure of the liquid in this part decreases. In the above equation, p is called the static pressure while $\frac{1}{2}\rho v^2$ is called the dynamic pressure.

2.6. VELOCITY OF EFFLUX OF A LIQUID — TORRICELLI'S THEOREM

Consider a tank containing a liquid of density ρ. Let the free surface of the liquid be at a height h above the level of a circular and sharp edge orifice O as shown in Fig. 2.6., such that the liquid is allowed to escape through O. Both the free surface of liquid and the orifice being exposed to the atmosphere have the pressure in them equal to the atmospheric pressure, say P. Also, if the tank be sufficiently wide, the velocity at the liquid surface may be taken to be zero. If v be the velocity of liquid at the level of the orifice, then considering a tube of flow from point A at the free surface to point O, we have by Bernoulli's theorem

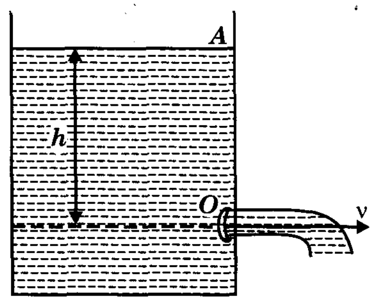


Fig. 2.4

total energy per unit mass at A = total energy per unit mass at O

$$\therefore \qquad \frac{p}{\rho}+0+hg=\frac{p}{\rho}+\frac{1}{2}v^2+0$$

or, $\quad v^2 = 2gh \qquad$ whence $v = \sqrt{2gh}$

This is the velocity of efflux at the orifice O. This result was first deduced by Torricelli (in 1644), and hence is known as Torricelli's theorem or the law of efflux, and may be stated as follows;

The velocity of efflux of a liquid at an orifice is equal to that attained by a body while falling freely from the surface of the liquid to the orifice.

Since, clearly if a body falls freely through the same height h under the action of gravity, then its velocity v is given according to the relation

$$v^2 - 0^2 = 2gh \qquad \Rightarrow \quad v=\sqrt{2gh}$$

That is, same as the relation obtained above. Since no liquid is perfectly free from internal friction or viscosity, hence this ideal velocity if seldom attained because of loss of kinetic energy of the liquid.

2.7. VENTURIMETER

A venturimeter is a gauge which is attached to a pipe to measure the rate of flow of liquid through the pipe and it is a practical application of Bernoulli's theorem.

Principle of the instrument is that when a liquid flows through a tube of varying areas of cross-section, the velocity and pressure vary along the length of the tube, the velocity being the greatest where the pressure is the least and vice-versa.

In its simplest form, a venturimeter is as shown in fig. 2.7. It consists of a constriction *B*, called the throat inserted in a pipeline with properly designed tapering both at inlet and outlet (*A* and *C* respectively) to ensure streamline flow and avoid turbulence. To measure the rate of flow of liquid, the venturimeter is inserted horizontally into the pipe line and the throat is kept at datum level.

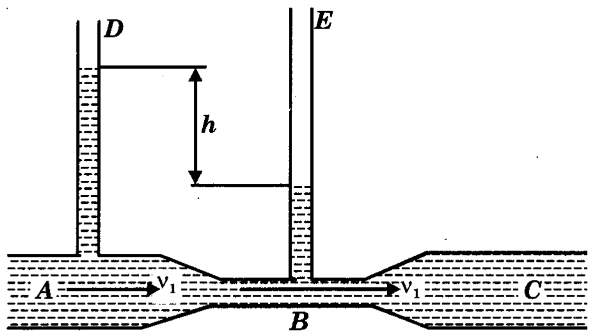


Fig: 2.7

When the liquid flows from *A* to *B*, owing to the decrease in the area of cross-section of the tube, the velocity of flow becomes greater. If v_1 is the velocity of flow at *A* of cross-sectional area A_1 and v_2 in the section *B* of cross-sectional area A_2, and if *V* is the volume of the liquid that passes in unit time across the tube, then according to the equation of continuity,

$$V = A_1 v_1 = A_2 v_2 \qquad (2.4)$$

As the mean height of the liquid in *A* and *B* is the same and its flow is horizontal, there is no change of gravitational potential energy. If p_1 be the pressure at *A* and p_2 be the pressure at *B*, and ρ be the density of the liquid, then according to Bernoulli's theorem,

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g}$$

or,
$$p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2)$$

Since $v_2 > v_1$, therefore form above equation, $p_1 > p_2$. So the pressure of liquid at the constriction is less than that in the main pipeline. Consequently, the level of liquid in the attached vertical side tubes is higher at the wide region than in the constricted region. If *h* be the difference in heights in the two vertical side tubes *D* and *E* at the wide and constricted region respectively, then

$$p_1 - p_2 = h\rho g \qquad (2.5)$$

Hence, from equations (2.4) and (2.5), we have

$$h\rho g = \frac{\rho}{2}\left(\frac{V^2}{A_2^2} - \frac{V^2}{A_1^2}\right)$$

$$v_1^2 = \frac{2(p_1 - p_2)}{p\left(\frac{A_1^2}{A_2^2} - 1\right)} = \frac{2gh}{\frac{A_1^2}{A_2^2} - 1} \qquad \text{(using 2.5)}$$

$$v_1 = A_2\sqrt{\frac{2gh}{A_1^2 - A_2^2}}$$

Thus, knowing h from the manometric reading and other quantities by auxiliary measurements, v_1 can be easily determined.

The volume of liquid flowing per second through any section of the pipe is,

$$V = A_1 v_1 = A_1 A_2\sqrt{\frac{2gh}{A_1^2 - A_2^2}}$$

Thus, the rate of flow V of the liquid through the pipe can be measured. The actual rate of flow, however, is less than this due to viscosity of the fluid and friction at the walls.

2.8. VISCOSITY

When a liquid moves slowly and steadily over a fixed horizontal surface, *i.e.,* when its flow is streamline, every layer of the liquid moves parallel to the fixed surface. Its layer in contact with the fixed surface is at rest and the velocity of every other layer increases uniformly and continuously with the distance from the fixed surface, *i.e.,* the greater the distance of a layer from the fixed surface, the greater is its velocity. Thus, there is a regular velocity gradient set up in the liquid, with the topmost layer moving the fastest.

Now consider the motion of two adjacent layers, say P and Q, of the liquid at distances x and $x + dx$ above the fixed horizontal surface, which are moving with the velocities v and $v + dv$ respectively (Fig. 2.8). As the upper layer Q is moving faster than the lower layer, the upper layer tends to increase the velocity of the lower layer by dragging the layer along with it, while the lower layer tends to decrease the velocity of the upper layer.

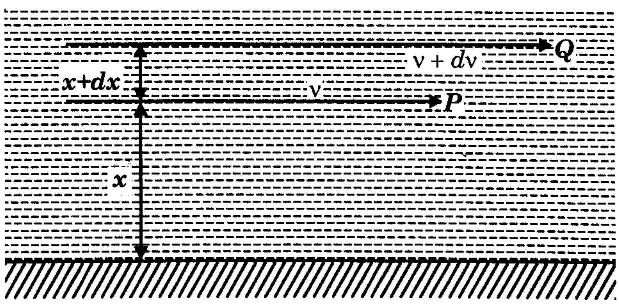


Fig: 2.8

The two layers together tend to destroy their relative motion as if there is some backward dragging force acting tangentially on the layers. Thus, to maintain the relative motion, an external force must be applied to overcome this backward drag. In the absence of any such outside force, the relative motion between the layers is destroyed and the flow of such a liquid ceases.

This property of a liquid by virtue of which it opposes the relative motion between its different layers is known as viscosity and the tangential force that tends to destroy the relative motion is called viscous force. Viscosity is also called the internal friction of the liquid.

Considering the two layers P and Q of the liquid, the separation between these layers is dx and the relative velocity between them is dv. The quantity dv/dx is called the ***velocity gradient***. According to Newton, the tangential backward dragging, or viscous force F, acting on a layer is proportional to its surface area A and velocity gradient dv/dx, *i.e.*

$$F \propto -A\frac{dv}{dx} \qquad \text{or} \qquad F = -\eta A\frac{dv}{dx} \tag{2.6}$$

Where, η is a constant, depending upon the nature of the liquid, and is called its coefficient of viscosity. The negative sign indicates that the viscous force acts opposite to the direction of flow of the liquid. Thus, it is obvious that the external force required in maintaining the relative velocity dv between the two layers is also equal to F, but opposite in direction.

If A=1 and dv/dx =1, then from (2.4), $F = \eta$

Hence, the ***coefficient of viscosity*** of a liquid may be defined as the tangential force acting per unit area between two parallel liquid layers, and required to maintain unit velocity gradient between these layers perpendicular to the direction of flow of the liquid.

The cgs unit of coefficient of viscosity is dyne-s/cm^2 or poise. In SI, the unit of η is N-s/m^2 or decapoise, where 1decapoise = 10 poise. The dimensions of η is $[ML^{-1}T^{-1}]$.

Viscosity in liquids corresponds to solid friction since like the latter, it also opposes the relative motion between two surfaces in contact i.e. the liquid layers. But, unlike solid friction, viscosity depends upon *(i)* The surface area of contact of the two layers *(ii)* The relative velocity between the two layers and *(iii)* the distance of separation between the two layers under consideration.

2.9. CRITICAL VELOCITY

The maximum velocity up to which the flow of a liquid is streamline is called its critical velocity. Above this value, the flow of a liquid ceases to be streamline and it becomes turbulent.

Osborne Reynold's showed by direct experiment that the critical velocity of flow of a liquid for long, narrow tubes depends upon *(i)* the coefficient of viscosity (η) of the liquid *(ii)* the density (ρ) of the liquid, and *(iii)* the radius (r) of the tube. The expression for the critical velocity (v_c) of a liquid can be derived by the method of dimensions.

Since v_c is found to depend upon h, ρ and r, therefore

$$v_c = k\,\eta^a \rho^b r^c$$, where k is a dimensionless constant.

Putting the dimensions of the various quantities involved, we have

$$[LT^{-1}] = [ML^{-1}T^{-1}]^a [ML^{-3}]^b [L]^c \quad \text{or,} \quad [LT^{-1}] = [M^{a+b} L^{-a-3b+c} T^{-a}]$$

According to the principle of homogeneity of dimensions, the dimensions on the two sides of the equation must be same, therefore

$$a+b = 0 \; ; \; -a-3b+c = 1 \;\text{ and } -a = -1$$

$$\therefore \quad a = 1, \quad b = -1 \quad \text{and } c = -1$$

Substituting the values of a, b and c, we have

$$v_c = k\,h^1 \rho^{-1} r^{-1}$$

or,
$$v_c = \frac{k\eta}{r\rho}$$

The constant k is called Reynold's number and its value is about 1000 for narrow tubes. It is a pure number and is independent of the system of units used for the measurement of various quantities. If k lies between 0(zero) and 2000, the flow of the liquid is streamline or laminar. For values of k above 3000, the flow of the liquid is turbulent and for values of k in the range 2000 to 3000, the flow of the liquid is unstable and may change from streamline to turbulent flow.

From the above equation, it is seen that the critical velocity of flow of a liquid is *(i)* directly proportional to its viscosity *(ii)* inversely proportional to its density and *(iii)* inversely proportional to the radius of the tube through which it flows.

Thus, it follows that narrow tubes and liquids of high viscosity and low density tend to promote orderly motion, whereas tubes of wide bores and liquids of low viscosity and high density lead to turbulence. Again, if a liquid is perfectly mobile, i.e., a liquid in which $\eta = 0$, then, $v_c = 0$; so that its flow would be turbulent and not orderly even for the smallest velocity and no matter how narrow the tube is. Thus, it is due to the viscosity of a liquid that a liquid can have a streamline flow.

2.10. POISEUILLE'S EQUATION FOR FLOW OF A LIQUID THROUGH A HORIZONTAL NARROW TUBE

Around 1940, the French physicists, Poiseuille derived a relation for the volume of a liquid flowing per second through a narrow horizontal capillary tube based on the following assumptions:

(i) The flow of the liquid is steady and streamline, with the streamlines everywhere parallel to the axis of the tube.

(ii) The pressure over any cross-section at right angles to the axis of the tube is constant, *i.e.*, there is no radial flow.

(iii) The velocity of the liquid layer in contact with the walls of the tube remains at rest and increases regularly towards the axis of the tube.

(iv) The tube is horizontal so that gravity does not affect the flow.

Consider a liquid flowing through a horizontal capillary tube of length l and radius r under a constant pressure difference p as shown in Fig. 2.9. Consider a cylindrical layer of the liquid, coaxial with the tube, of radius x and thickness dx. The liquid on the inner side of this cylindrical layer is moving faster while that on the outer side is moving slower. Let the velocity of the liquid at a distance x from the axis of the tube is v and at a distance $x+dx$ is $v - dv$, so that dv/dx is the velocity gradient. Therefore, the tangential force exerted by the outer layer on the inner layer opposite to the direction of flow, in accordance with Newton's law of viscous flow, is given by

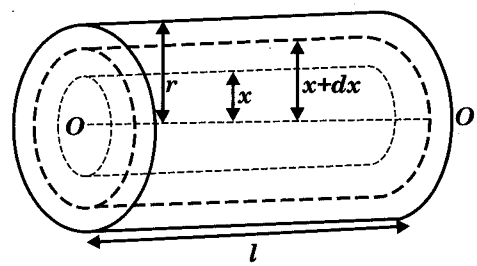


Fig: 2.9

$$F = -\eta\,(2\pi x l)\frac{dv}{dx}$$

Where, $2\pi\, xl$ is the surface area of the cylinder of radius x, and η is the coefficient of viscosity of the liquid.

The force driving the liquid forward due to the pressure difference p at the two ends of the cylinder of radius $x = p \times \pi x^2$.

When the flow of the liquid is steady, there is no acceleration of the liquid and thus, we have

$$-\eta\,(2\pi x l)\frac{dv}{dx} = p\pi x^2$$

or,

$$dv = -\frac{p}{2\eta l}\,x\,dx$$

Integrating both sides, we get $v = -\frac{p}{2\eta l}\frac{x^2}{2} + C$, where C is the constant of integration

When $x = r$, $v = 0$; since the velocity of the layer in contact with the walls of the tube is zero.

$$\therefore\; 0 = -\frac{p}{2\eta l}\frac{r^2}{2} + C \qquad \text{or} \qquad C = \frac{p}{2\eta l}\frac{r^2}{2}$$

Hence, $$v = -\frac{p}{2\eta l}\frac{x^2}{2} + \frac{p}{2\eta l}\frac{r^2}{2}$$

or, $$v = \frac{p}{4\eta l}(r^2 - x^2)$$

This expression gives the velocity of flow of the liquid at a distance x from the axis of the tube. The velocity distribution curve of the advancing liquid in the tube is a parabola; the velocity increases from 0 (zero) at the walls ($x=r$) of the tube to a maximum at its centre ($x=0$).

Now considering the cylindrical layer of radius x and thickness dx, the volume of the liquid flowing per second through the cylinder given by,

dV= velocity × area of cross-section of the cylinder

$$= v(2\pi x\, dx)$$

Therefore, the total volume of the liquid flowing per second through entire tube is

$$V = \int_0^r 2\pi v x\, dx$$

or, $$V = \int_0^r \frac{p}{4\eta l}(r^2 - x^2).\,2\pi x\, dx = \frac{\pi p}{2\eta l}\int_0^r (xr^2 - x^3)\, dx$$

$$= \frac{\pi p}{2\eta l}\left[\frac{r^2x^2}{2} - \frac{x^4}{4}\right]_0^r \quad \text{or} \quad = \frac{\pi p}{2\eta l}\left[\frac{r^4}{2} - \frac{r^4}{4}\right]$$

or, $$V = \frac{\pi p r^4}{8\eta l}$$

This is known as Poiseuille's formula. This formula is true only for the streamline flow of a liquid. Knowing the values of p, r, l and V, the coefficient of viscocity of the liquid can be determined.

The flow of a liquid is streamline only when the average velocity of flow of the liquid is less than the critical velocity, and since the critical velocity of flow is inversely proportional to the radius of the tube, therefore, the above formula fails in case of tubes of wide bores. Moreover, Poiseuille's formula has been derived on the basis that the pressure difference across the ends of the tube is used to overcome the viscous forces and no part of it is spent in imparting any kinetic energy to the liquid. Thus the formula is true for liquids (or fluids) for which the kinetic energy of the liquid is negligible i.e. the velocity is small, or in other words, the flow is streamline.

2.11. DETERMINATION OF COEFFICIENT OF VISCOSITY BY USING POISEUILLE'S FORMULA

From Poiseuille's formula, the coefficient of viscosity of a liquid for streamline flow of the liquid through a horizontal capillary tube is given by,

$$\eta = \frac{\pi p r^4}{8Vl} \tag{2.7}$$

Where V is the volume of liquid flowing per second through the tube, p is the pressure difference across the ends of the tube, l the length and r the radius of the tube.

Experimental Arrangement: The apparatus for the determination of coefficient of viscosity by Poiseuille's formula is shown in Fig.2.10. It consists of a long narrow capillary tube T of uniform bore which is fitted horizontally and having its ends inserted into two brass unions B and C and fitted with a rubber bung at each end. The capillary tube is connected to an out-flow tube D through C where it can be collected in a measuring jar, and to a water (experimental liquid) vessel A through B by using rubber tubes. A pinch cock K is used to regulate the flow of the liquid through the capillary tube.

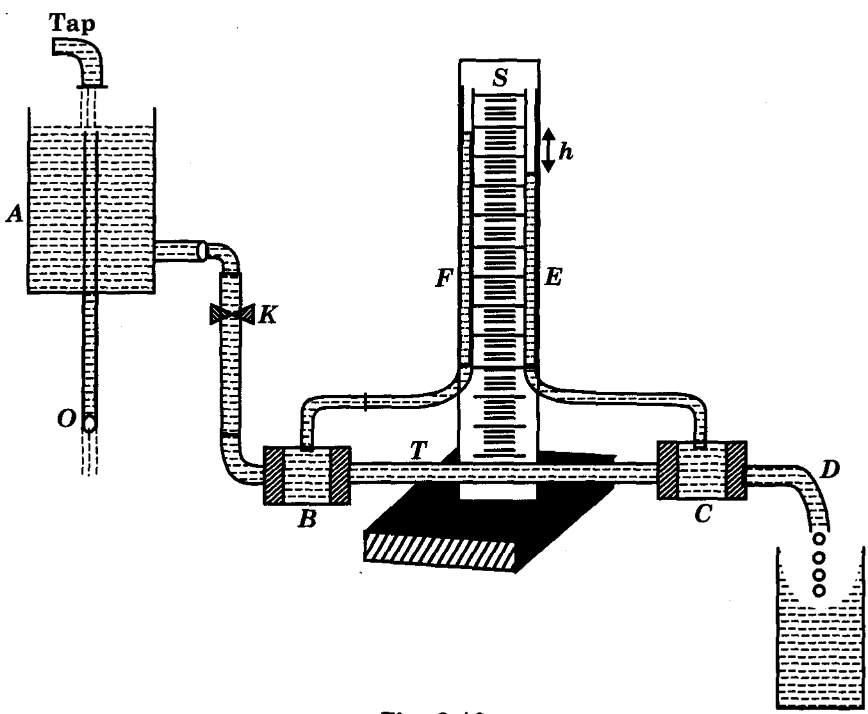


Fig: 2.10

The unions are connected to the two limbs of a manometer M by means of rubber tubes, and the pressure difference between the ends of the capillary tube T is noted by the difference in the levels of the liquid in the two limbs of the manometer. The vessel A, in which the tap water is coming, is fitted with an overflow tube O, so that a constant head of water may be maintained in the system.

Procedure: with the help of the pinch cock, a steady flow of water is maintained through the capillary tube such that water leaves the tube in a slow trickle. When the flow becomes steady, this may be noted by noting that the difference in the levels of liquid in the limbs of the manometer is constant, the emerging liquid is collected in a graduated jar for a certain interval of time. Thus, the volume of liquid flowing per second V is found out for a constant difference of levels h in the manometer. The experiment is repeated a number of times for different values of h. The pressure difference is directly read from the manometer and is equal to $h\rho g$. The radius of the capillary tube r can be determined by using a high power travelling microscope and its length l can be measured by a scale. Thus knowing the various quantities, the value of η can be determined from equation (2.5).

Precautions: *(i)* since r is involved with fourth power in the expression for$h$, extra care has to be taken while measuring the diameter of the capillary tube.

(ii) The pressure difference should be kept small to obtain streamline flow.

(iii) The bore of the tube must be narrow as the formula is applicable only for tubes of narrow bores.

(iv) The tube should be long enough so that any non-uniformity of flow at the inlet be negligible.

(v) The tube should be placed horizontally to avoid any effect of gravity.

2.12. STOKES' LAW: TERMINAL VELOCITY

When a body falls through a viscous medium, it carries along with it the fluid in contact and thus tends to create a relative motion between the layers in contact with it and the layers below it; whereas the layer at an infinite distance from it is at rest. Thus a frictional retarding force, due to the viscosity of the medium, acts opposing the relative motion between the layers of the liquid which in turn is experienced by the body in motion. The opposing force or viscous force increases with the velocity of the body, until, in the case of small bodies, it becomes just equal to the driving force, and the body then attains a constant velocity, called its ***terminal velocity***. Stokes showed that for a small spherical body of radius r moving through a viscous medium of coefficient of viscosity η, the viscous drag F acting on the body is given by,

$$F = 6\pi\eta rv \tag{2.6}$$

The above relation is known as Stokes' law, which may be deduced as follows, by the method of dimensions. (The rigorous deduction of Stokes' law is beyond the scope of this book).

Consider a small spherical body of radius r moving with a velocity v through a viscous medium of coefficient of viscosity h. According to Stokes, the viscous retarding force F acting on the body depends upon— *(i)* the coefficient of viscosity of the medium, *(ii)* the radius of the spherical body and *(iii)* the velocity of the body.

Let $F = k\,\eta^a\,r^b\,v^c$

Where k is a dimensionless constant of proportionality and a, b, c are arbitrary constants to be determined.

Putting the dimensions of the various terms in the above equation, we have

$$[MLT^{-2}] = [ML^{-1}T^{-1}]^a\,[L]^b\,[LT^{-1}]^c$$
$$= [M^aL^{-a+b+c}T^{-a-c}]$$

Whence, by principle of homogeneity of dimensions,

(i) a = 1 (ii) –a + b +c = 1 (iii) –a –c = –2

On solving, we get a = 1, b = 1, c = 1

$\therefore$ $F = k\eta\,r\,v$

The value of k was found by Stokes to be 6π.

Hence, $F = 6\pi\eta\,r\,v$

If the density of the sphere be ρ, its weight W = volume × density × acceleration due to gravity

$$= \frac{4}{3}\pi r^3\rho g$$

And, if σ be the density of the medium, then the upthrust on the body due to the displaced medium, *i.e.* buoyant force,

T = weight of the liquid displaced

$$= \frac{4}{3}\pi r^3\sigma g$$

$\therefore$ resultant downward force on the body

$$= \frac{4}{3}\pi r^3\rho g - \frac{4}{3}\pi r^3\sigma g$$

$$= \frac{4}{3}\pi r^3(\rho - \sigma)g$$

When the body attains terminal velocity (v_T), the net vertical force acting on the body becomes zero, i.e. the resultant downward force becomes equal to the retarding force acting on the body.

Thus, $$6\pi\eta r v_T = \frac{4}{3}\pi r^3(\rho - \sigma)g$$

$$\Rightarrow \quad v_T = \frac{2}{9}\frac{r^2(\rho - \sigma)g}{\eta}$$

From the above equation, it is seen that the terminal velocity of the spherical body falling through the viscous medium is:

(a) directly proportional to the square of the radius of the spherical body. Thus larger bodies have larger terminal velocities and vice-versa. This explains why larger rain drops hurts more when they fall on a person than smaller drops.

(b) directly proportional to the difference in the densities of the body and the medium respectively. If $\rho > \sigma$, then v_T is positive. In this case, the body will fall through the viscous medium. But if $\rho < \sigma$, then v_T is negative; in that case, the body will rise through the medium instead of falling through it. For example, rain drops fall through air, while an air bubble inside a liquid rises up through the liquid.

(c) inversely proportional to the coefficient of viscosity of the liquid. Thus, larger the viscosity of the medium, smaller is the terminal velocity of the body falling through the medium and vice-versa.

In arriving at the above results, Stokes made a number of assumptions which are as follows:

- The medium through which the body is falling is infinite in extent.
- The medium is a continuous, homogenous fluid and the diameter of the spherical body is much larger than the size of the molecules of the medium.
- The spherical body is perfectly rigid and smooth.
- The velocity of motion of the body is less than the critical velocity of the medium and that there are no eddy currents or waves which are set up in the medium due to the motion of the body through it.

This case which have been discussed find useful applications in determining the coefficient of viscosity of a liquid, the radius of small spherical objects like rain drops etc., and the charge of electrons by Millikan's oil drop experiment. However, since a medium of infinite extent is not possible in such experiments, corrections are to be made for the finite boundaries of the vessel — the so called, ***wall-effect correction*** and the ***bottom correction. Ladenburg*** made the following corrections for the 'wall-effect' and the 'bottom-effect' respectively,

$$v_\infty = v\left(1 + 2.4\frac{r}{R}\right) \text{ and } v_\infty = v\left(1 + 3.3\frac{r}{D}\right)$$

Where, v is the observed velocity of the sphere of radius r, v_∞ is the velocity of the sphere in the medium of infinite extent, R is the radius of the vessel and D is the full depth of the liquid column in the vessel. Combining the two corrections, therefore, we have the following relation for η, *viz.*,

$$\eta = \frac{2}{9}\frac{r^2 g(\rho - \sigma)}{v\left(1 + 2.4\frac{r}{R}\right)\left(1 + 3.3\frac{r}{D}\right)}$$

2.13. EFFECT OF TEMPERATURE AND PRESSURE ON VISCOSITY:

Temperature has a very marked effect on the viscosity of fluids. While the viscosity of liquids decreases rapidly with temperature that of gases increases with the increase in temperature.

For example, the viscosity of water at 80°*C* is found to be only one-third of its value at 10°*C*. Although various empirical relations have been suggested from time to time, however, no satisfactory relation could be suggested to express the relationship between viscosity and temperature. Within a narrow range of temperature, ***Slotte*** gave the following empirical formula for liquids.

$$\eta_t = \frac{\eta_0}{1 + \alpha t + \beta t^2}$$

Where α and β are constants and η_t and η_0 are the viscosities at t^0C and 0^0C respectively. Other empirical relations were also given by other workers as follows:

$$\log \eta_T = \alpha + \frac{\beta}{T} \qquad \text{and} \qquad \eta_t = \frac{A}{(1+Bt)^c}$$

Where A, B and C are constants and η_T is the viscosity at T K. The second relation, however, is not applicable to oils.

Andrade, based on certain assumptions, gave a satisfactory theory for liquids. He put forward the relation, $\eta\rho^{\frac{1}{3}} = Ae^{\frac{cp}{T}}$ for variation of viscosity of liquids with absolute temperature; *A*, *C* being constants and ρ the density of the liquid. This relation is in fair agreement with experimental results for many liquids except water and some alcohols.

According to the kinetic theory, the viscosity of a gas should be directly proportional to the square root of the absolute temperature and is given by the relation, $\eta_T = A\eta_0 T^{1/2}$, where *A* is a constant, η_0 is the coefficient of viscosity at 0°*C* and *T* is the absolute temperature. In practice, this holds only approximately, because real gases are not ideal.

A more correct relation was given by Sutherland as follows:

$$\frac{\eta_T}{\eta_0} = \frac{273 + C}{T + C}\left(\frac{T}{273}\right)^{1.5}$$

Where η_T and η_0 are the viscosities at *T K* and 0^0C and *C* is a constant known as *Sutherland constant* which is different for different gases. This formula agrees well with experimental data.

The effect of pressure on viscosity is small for mobile liquids. For liquids having large values of viscosity, however, the effect is considerable. For all liquids, except water, the viscosity increases with increase of pressure. The viscosity of water decreases with the increase of pressure up to a few hundred atmospheres.

As per the kinetic theory of gases, the viscosity of gases is independent of the pressure over a wide range at ordinary pressures. At low pressures, $\eta \propto p$, while at high pressures (~ 500 atmospheres), η increases by 25-50 % of its value at ordinary pressures.

2.14 SOLVED EXAMPLES

1. A metal plate 10sq.cm rests on a 2mm thick castor oil layer. Calculate the horizontal force needed to move the plate with a speed of 3cms^{-1}. Coefficient of viscosity of castor oil is 15poise.

Solution: Here, Area, A = 10cm^2; dv = 3cms^{-1}; dx = 2mm = 0.2cm

Coefficient of viscosity, h = 15poise

Force required in moving the plate is given by, $F = \eta V \frac{dv}{dx}$

= 15 × 10 × (3/0.2)

= 2250 dyne

2. Water is flowing with a speed of 50cm/s through a pipe of diameter 3mm. Calculate the value of Reynolds's number and characterizes the flow. Coefficient of viscosity of water is 1 centipoise.

Solution: Here, density of water, r = 1g/cc; coefficient of viscosity, h = 10^{-2}poise; v = 50cm/s;

d = 3mm = 0.3cm

Reynolds's number R is given by, $R = \frac{\rho v d}{\eta}$

$$\therefore \quad R = \frac{1 \times 50 \times 0.3}{10^{-2}} = 1500$$

Since R = 1500, hence the flow is streamline.

3. Water is flowing through a horizontal pipe 8cm in diameter and 4 kilometers in length at the rate of 20lit/s. Assuming only viscous resistance, calculate the pressure required to maintain the flow in terms of mercury column. Coefficient of viscosity of water is 0.001 Pa-s.

Solution: Here, r = 8/2 = 4cm = 0.04m; l = 4km = 4000m; V = 20lit/s = 20×10^{-3}m^{3}s^{-1};

d = 3m

η = 0.001Pa

$$\text{As,} \quad V = \frac{\pi p r^4}{8 p \eta l} \quad \text{or,} \quad p = \frac{8 V \eta l}{\pi r^4}$$

$$\therefore \quad p = \frac{8 \times 20\, 10^{-3} \times 0.001 \times 4000}{3.14 \times 0.04^4} = 7.954 \times 10^4\ Pa$$

∴ Height of mercury column for pressure difference p will be,

$$\therefore \quad h = \frac{p}{\rho g} = \frac{7.954 \times 10^{-4}}{(13.6 \times 10^3) \times 9.8} = 0.5968m = 59.68\text{cm}$$

4. At what speed will the velocity of a stream of water be equal to 20cm of mercury column. Take $g = 10ms^{-2}$.

Solution: Here, velocity head = 20cm 0f Hg = 20×13.6cm of water

$$\text{Now, velocity head} = \frac{v^2}{2g}$$

$$\therefore \quad 20 \times 13.6 = \frac{v^2}{2 \times 1000}$$

$$\Rightarrow \quad v = \sqrt{20 \times 13.6 \times 2 \times 1000} = 737.56\text{cm/s} = 7.3756\text{m/s}$$

5. What should be the minimum velocity of water in a tube of diameter 2.0 cm so that the flow is turbulent? The viscosity of water is 0.001NN-sm^{-2}.

Solution: Minimum value of Reynolds's number for the flow to be turbulent, $N_R = 3000$.

Here, D = 2cm = 0.02m; $\rho = 10^3$ kgm^{-3}; η = 0.001 N-sm^{-2}

$$\text{Now,} \quad v_c = \frac{N_R \eta}{\rho D} = \frac{3000 \times 0.001}{10^3 \times 0.02}$$

$$= 0.15 \text{ ms}^{-1}$$

6. Water flows through a horizontal pipe of which the cross-section is not constant. The pressure is 1cm of mercury where the velocity is 0.35 ms^{-1}. Find the pressure at a point where the velocity is 0.65 ms^{-1}.

Solution: Here, at one point, p_1 = 1cm of Hg = 0.01 m of Hg

$$= 0.01 \times (13.6 \times 10^3) \times 9.8 \text{ Pa}$$

$$v_1 = 0.35 \text{ ms}^{-1}$$

At another point, $v_2 = 0.65$ ms^{-1}

If p_2 be the pressure at this point, then according to Bernoulli's theorem

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

$$\text{or,} \quad p_2 = p_1 - \frac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

$$= 0.01 \times 13.6 \times 10^3 \times 9.8 - \frac{1}{2} \times 10^3 \times [(0.65)^2 - (0.35)^2]$$

$$= 1182.8 \text{ Pa} = \frac{1182.8}{9.8 \times 13.6 \times 10^3}$$

$$= 0.00887 \text{ m of Hg}$$

7. The diameter of a pipe at two points where a venturimeter is connected is 8cm and 5cm and the difference of levels in it is 4cm. Calculate the volume of water flowing through the pipe per second.

Solution: Here, r_1 = 8/2 = 4 cm; r_2 = 5/2 = 2.5 cm; h = 4cm

Here, $a_1 = \pi r_1^2 = \pi(4)^2 = 16\pi cm^2$ and $a_2 = \pi r_2^2 = \pi(2.5)^2 = 6.25\pi cm^2$

Here, density of water, $\rho = 1$ g/cc, so that the volume of water flowing through the pipe per second is,

$$V = A_1 A_2 \sqrt{\frac{2gh}{A_1^2 - A_2^2}}$$

$$= 6.25\pi \times 16\pi \sqrt{\frac{2 \times 980 \times 4}{(16\pi)^2 - (6.25\pi)^2}}$$

$$= 1189 \text{ cc/s}$$

8. Find the velocity of efflux of water from an orifice near the bottom of a tank in which pressure is 500gf/cm^2 above atmospheric.

Solution: Here, pressure at the orifice, $P = 500\text{gf/cm}^2 = (500/1000) \times 9.8 \times 100^2 \text{ Nm}^{-2}$

$$= 0.01 \times (13.6 \times 10^3) \times 9.8 \text{ Pa}$$

$$= 500 \times 98 \text{ Nm}^{-2}$$

Let h be the depth of the orifice below the surface of the water in the tank.

As, $P = h \rho g$

$$\therefore \quad h = P/\rho g = \frac{500 \times 98}{10^3 \times 9.8} = 5m$$

Hence, the velocity of efflux, $v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 5} = 9.893 ms^{-1}$

9. The velocity of a small ball of mass 10g and density 7.8g/cc when dropped in a container containing glycerin becomes constant after sometime. If the density of glycerin is 1.3g/cc, what is the viscous force acting on the ball?

Solution: Here, m = 10g; $\rho = 7.8$g/cc; $\alpha = 1.3$g/cc.

Weight of the ball = mg = 10×980 dyne

Volume of the ball, $V = m/\rho_1 = 10/7.8$ cc

Buoyant force on the ball = weight of displaced liquid

$$= \frac{m}{\rho} \times \sigma \times g = \frac{10}{7.8} \times 1.3 \times 980 \text{ dyne}$$

$\therefore$ Viscous force acting on the ball = net weight of the ball

$$= 10 \times 980 - \frac{10}{7.8} \times 1.3 \times 980$$

$$= 8.166 \times 10^3 \text{ dyne}$$

10. Twenty seven identical drops of water are falling down vertically in air each with a terminal velocity of 0.15 ms^{-1}. If they combine to form a single drop, what is the terminal velocity of the big drop?

Solution: Let the radius of each small drop be r.

Terminal velocity of each small drop, $v = 0.15 ms^{-1}$

Now, $$v = \frac{2r^2(\rho-\sigma)g}{9\eta} \qquad ...(i)$$

Let R be the radius of the bigger drop.

Now, Volume of one big drop = volume of 27 small drops

i.e, $$\frac{4}{3}\pi R^3 = 27 \times \frac{4}{3}\pi r^3$$

or, $$R = 3r$$

The terminal velocity of the bigger drop, $$v' = \frac{2R^2(\rho-\sigma)g}{9\eta} \qquad ...(ii)$$

Dividing (*ii*) by (*i*), we get

$$\therefore \quad \frac{v'}{v} = \frac{R^2}{r^2} = \left(\frac{R}{r}\right)^2 = 9$$

$$\therefore \quad v' = 9v = 9 \times 0.15 = 1.35\text{ms}^{-1}$$

11. A spherical ball of radius 10^{-4}m and density 10^4kgm^{-3} falls freely under gravity through a vertical distance h before entering a tank of water. If after entering the water, the velocity of the ball does not change, find h. The coefficient of viscosity of water is 9.8×10^{-6} N-sm^{-2}

Solution: The velocity attained by the ball after falling freely through a height h is,

$$v = \sqrt{2gh} \qquad or, \qquad h = \frac{v^2}{2g} \qquad ...(i)$$

Terminal velocity of each small drop, $v = 0.1$

Where, v is the velocity with which the ball falls through and is its terminal velocity.

Now, terminal velocity of the ball is,

$$v = \frac{2r^2(\rho-\sigma)g}{9\eta}$$

Here, $\rho = 10^4$kgm–3; σ = *density of water* = 10^3kgm^{-3}; $\eta = 9.8\times10^{-6}$N-sm^{-2};

$r = 10^{-4}$m

$$\therefore \quad v = \frac{2\left(10^4-10^3\right)\times 9.8\times10^{-8}}{9\times9.8\times10^{-6}} = 20\text{ms}^{-1}$$

Substituting the value of v in eqn. (*i*), we get

$$h = \frac{400}{2\times9.8} = 20.4m$$

11. Calculate the rate of flow of glycerin of density 1.25×10^3 kgm^{-3} through the conical section of a pipe if the radii of its ends are 0.1m and 0.04m, and the pressure drop across its length is 10Nm^{-2}.

Solution: According to Bernoulli's theorem,

$$v = \frac{2r^2(\rho-\sigma)g}{9\eta}$$

$$\frac{p_1}{\rho}+gh_1+\frac{1}{2}v_1^2 = \frac{p_2}{\rho}+gh_2+\frac{1}{2}v_2^2$$

For horizontal flow, $h_1 = h_2$

$$\therefore \quad \frac{p_1}{\rho}+\frac{1}{2}v_1^2 = \frac{p_2}{\rho}+\frac{1}{2}v_2^2$$

or, $$\frac{(p_1 - p_2)}{\rho} = \frac{1}{2}(v_2^2-v_1^2)$$

$$\therefore \quad (v_2^2-v_1^2) = \frac{2(p_1 - p_2)}{\rho}$$

Now, $p_1 - p_2 = 10Nm^{-2}; \rho = 1.25\times10^3 kgm^{-3}$

$$\therefore \quad (v_2^2-v_1^2) = \frac{2\times10}{1.25\times10^3} = 16\times10^{-3} \qquad ....(i)$$

According to the equation of continuity, $a_1v_1 = a_2v_2$

$$\therefore \quad \frac{v_1}{v_2} = \frac{a_2}{a_1} = \frac{\pi r_2^2}{\pi r_1^2} = \frac{r_2^2}{r_1^2}$$

$$= \frac{0.04^2}{0.1^2} = 1.6\times10^{-2}$$

or, $$v_1 = (16\times10^{-2})\, v_2$$

Hence, from eqn. (*i*), we have

$$= v_2^2 - 256\times10^{-4} v_2^2 = 16\times10^{-3}$$

$$v_2^2\left[1-\frac{256}{10^4}\right] = 16\times10^{-3}$$

$$v_2^2\left[\frac{9774}{10^4}\right] = 16\times10^{-3}$$

$$v_2^2 = \frac{160}{9774} \text{ or, } v_2 = 0.13ms^{-1}$$

Hence, the rate of flow of glycerin $= a_2v_2 = \pi r_2^2 v_2 = 3.14\times(0.04)^2\times0.13$

$$= 6.53\times10^{-4} m^3s^{-1}$$

2.15 SAMPLE QUESTIONS

1. Distinguish between streamline and turbulent flow and explain the significance of Reynolds's number.
2. What is critical velocity and Reynolds's number? Obtain dimensionally the relation between the critical velocity and this number for a liquid flowing through a capillary tube.
3. Explain the terms streamline flow and turbulent flow. Derive the equation of continuity for the steady flow of an incompressible liquid.
4. Define coefficient of viscosity and the poise. State its SI unit and also find its dimensions.
5. State and prove Bernoulli's theorem for fluids and give the assumptions used in deriving it.
6. State Bernoulli's theorem. Prove that the total energy possessed by a flowing ideal liquid is conserved, stating clearly the assumptions made.
7. What are the various types of energy possessed by a flowing liquid? Derive the expression for the energy per unit mass of a flowing liquid.
8. State and prove Stoke's law and hence find an expression for the terminal velocity of a spherical body falling through a viscous medium. Discuss the factors on which the terminal velocity depends.
9. Prove that the velocity of efflux of an ideal liquid through an orifice is equal to the velocity attained by a freely falling body from the surface of the liquid to the orifice.
10. What do you mean by terminal velocity? Derive an expression for the same.
11. Derive Poiseuille's formula for the steady flow of an incompressible viscous liquid through a horizontal capillary of uniform cross-section.
12. Describe an experiment o the basis of Poiseuille's formula to determine the viscosity of a liquid.
13. A sphere of radius r travelling through a viscous liquid of density ρ with a velocity v experiences a retarding force F given by the relation,

    $$F = k\, r^a \rho^b (\eta v)^c$$

    Where k is a constant and η is the coefficient of viscosity of the liquid. Use dimensional analysis to determine a, b and c.
14. Discuss the effect of temperature and pressure on the viscosity of fluids.
15. Show how the Bernoulli's theorem is applied to measure the rate of discharge of water through the city water mains.

1.29 SAMPLE PROBLEMS

1. 1. A liquid is flowing through a horizontal pipe line of varying cross-section. At a certain point, the diameter of the pipe is 5cm and velocity of flow of liquid is $2.5 cms^{-1}$. Calculate the velocity of flow at another point where the diameter is 1cm. (**Ans.** 62.5 cms^{-1})
2. Water flows through a hose pipe whose internal diameter is 2cm at a speed of $1 ms^{-1}$. Calculate the diameter of the nozzle if the water is to emerge at a speed of $4 ms^{-1}$. (**Ans.** 1cm)
3. At what speed will the velocity head of water be equal to 20cm? (**Ans.** 1.98 ms^{-1})
4. A tank containing water has an orifice 10 metre below the surface of water in the tank. Assuming that there is no wastage of energy, find the speed of discharge. (**Ans.** 14 ms^{-1})
5. A capillary tube of 1mm diameter and 15cm long is fitted horizontally to a vessel kept full of alcohol of density 0.8 gcm^{-3}. The depth of the centre of capillary tube below the free surface of alcohol is 25cm. If the viscosity of alcohol is 0.12poise, find the amount that will flow in five minutes. (**Ans.** 6.408 g)
6. A pipe is running full of water. At a certain point *A* it tapers from 50cm diameter to 15 cm at point *B*. The pressure difference between *A* and *B* is 10cm of water column. Find the rate of flow of water through the pipe, the pipe being horizontal. (**Ans.** 3.5×10^4 cm^3s^{-1})
7. Water from a tap emerges vertically downward with an initial speed of $1.0 ms^{-1}$. The cross-sectional area of the tap is $10^{-4} m^2$. Assume that the pressure is constant throughout the stream of water, and that the flow is steady, what is the cross-sectional area of the stream 0.15m below the tap? (**Ans.** 5.0×10^{-5} m^2)
8. The pressure difference between two points along a horizontal pipe through which water is flowing is 1.4 cm of mercury. If due to non-uniform cross-section, the speed of flow of water at the point of greater cross-section is $60 cms^{-1}$, calculate the speed at the other end. (**Ans.** 63 cms^{-1})
9. What volume of water will escape per minute from a tank through an opening 1cm in diameter and 5.1m below the level of water? (**Ans.** 4.7 m^3/min)
10. Water is conveyed through a horizontal tube 8cm in diameter and 4km in length at a rate of 20lit/s. Assuming only viscous resistance, calculate the pressure required to maintain the flow. (η of water = 0.01poise). (**Ans.** 7.95×10^5 $dynecm^{-2}$)
11. A venturimeter is connected to a horizontal main of radius 20cm. If the radius of the throat of the venturimeter is 15cm and the difference of water level is venturimeter tubes is 10cm, calculate the rate of flow of water per hour through the main. (**Ans.** 43.11×10^4 lit/hour)
12. If a venturimeter is connected at two points where the diameters of the pipe are 8cm and 5cm and the pressure difference between two points is shown to be 6cm of water column, determine the rate of at which the water is flowing through the pipe. (**Ans.** 2.31×10^{-3} m^3s^{-1})
13. Calculate the velocity of efflux of water from a tank in which the pressure is $980 Nm^{-2}$ above the atmospheric pressure. (**Ans.** 1.4 ms^{-1})
14. Find the terminal velocity of an oil drop of density 0.95 g/cc and radius 10^{-4} cm falling through air of density 0.0013 g/cc, if the viscosity of air is 18.1×10^{-5}poise and g = 980 cms^{-2}. (**Ans.** 1.33×10^{-2} cms^{-1})
15. A flat plate of area $2\times10^{-2} m^2$ is separated from a large flat surface by a film of oil of uniform thickness 1.5×10^{-3}m and viscosity 2N-s/m^2. Determine the force required to slide the plate over the surface at a velocity of $4.5\times10^{-2} ms^{-1}$. (**Ans.** 1.2N)

16. A spherical glass ball of mass 1.34×10^{-4}kg and diameter 4.4×10^{-3}m takes 6.4s to fall steadily through a height of 0.381m inside a large volume of oil of specific gravity 0.943. Find the viscosity of the oil. (**Ans.** 0.365Nsm^{-2})

17. A soap bubble of radius 4cm and surface tension 30dyne/cm is blown at the end of a capillary tube of length 10cm and internal diameter 0.2cm. If η = 1.85×10^{-4}poise, find the time for the reduction of the bubble to 2cm radius. (**Ans.** 4min-56s)

18. Water flows through a horizontal tube of length 20cm and internal radius 0.081cm under a constant head of the liquid 20cm high. In 12 minutes, 864 cc of liquid issue from the tube. Find the viscosity of water and verify that the flow is streamline. (**Ans.** 0.0138poise)

19. Two drops of water of the same size are falling through air wit terminal velocity 1ms^{-1}. If the two drops combine to form a single drop, calculate its terminal velocity. (**Ans.** 1.588 ms^{-1})

20. An air bubble of 1cm radius rises through a long narrow column of liquid of density 1.47×10^3 kgm^{-3} with a steady velocity of 0.21cms^{-1}. Find the viscosity of the liquid, neglecting the density of air (g = 9.8ms^{-2}). (**Ans.** 1.52×10^2 kgm^{-1}s-1)

21. The flow rate of water from a tap of diameter 1.50 cm is 3 lit per minute. Calculate the Reynolds's number and characterize the flow (η for water = 10^{-3} Pa-s).
(**Ans.** 4242.4, turbulent)

22. Eight rain drops of radius 1mm each falling downwards with a terminal velocity of 5cms^{-1} coalesces to form a bigger drop. Find the terminal velocity of the bigger drop.
(**Ans.** 20 cms^{-1})

23. The terminal velocity of a copper ball of radius 2.0mm falling through a tank of oil at 20°C is 6.5cms^{-1}. Compute the viscosity of the oil at 20°C. Density of oil is 1.5×10^3 kgm^{-3}, density of copper is 8.9×10^3 kgm^{-3}. (**Ans.** 0.99 Pa-s)

24. The flow rate of water from a tap of diameter 1.25cm is 0.48lit/min. The coefficient of viscosity of water is 10^{-3}Pa-s. After sometime, the flow rate is increased to 3lit/min. Characterize the flow for both the flow rates. (**Ans.** Streamline, Turbulent)

25. What should be the average velocity of water in a tube of radius 0.0005 m so that the flow is just turbulent? The viscosity of water is 0.001Pa-s. (**Ans.** 0.3 ms^{-1})

26. Calculate the total energy possessed by 1kg of water at a point where the pressure is 20g.wt./sq.mm, velocity is 0.1ms^{-1} and the height is 50cm above the ground level. (**Ans.** 200.905 J)

27. Water flows at the rate of 4lit/s through an orifice at the bottom of a tank which contains water 720cm deep. Find the rate of escape of water from the orifice if an additional pressure of 16 kg/cm^2 is applied at the surface of water. (**Ans.** 19.28litres/s)

28. A cubical vessel of height 1m is full of water. What is the work done in pumping water out of the vessel? (**Ans.** 4900 J)

29. A large tank filled with water to a height *h* is said to be emptied through a small hole at the bottom. Find the ratio of time taken for the level of water to fall down from *h* to *h/2* and *h/2* to zero. (**Ans.** $\sqrt{2}$ -1)

30. The flow of blood in a large artery of an anaesthetized dog is diverted through a venturimeter. The wider part of the meter has a cross-sectional area equal to that of the artery which is 8mm^2. The narrower part has an area of 4mm^2. The pressure drop in the artery is 24Pa. What is the speed of the blood in the artery?(density of blood = 1.06×10^3 kgm^{-3}) (**Ans.** 0.123 ms^{-1})

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INTRODUCTION TO

FLUID MECHANICS

The present book has been written taking into account the Syllabus of the North Eastern Hill University for the first year students, and subsequently with the introduction of the semester system, for the students of the first semester. Indeed, there is quite a number of books that which are already available on the topics covered, but it is a humble beginning on my part to try and present the subject matter in a simple, concise manner and taking into account the fact that, the average student needs to have a thorough understanding of the basic principles and concepts before he/she can really have an interest for the subject matter. Although the topics have been covered in most of the syllabi of different boards of Higher Secondary Education, however, particular care have been taken to make the treatment clear and comprehensible to the slightest detail, keeping in mind the fact that the average student may not have been able to grasp the subject matter completely and thoroughly by simply learning about it in the earlier class because of time constrain and many other factors. The mathematical treatment involved in the different sections is, therefore, given in detail and in a way to make the reading easy to understand and follow. In fact, any sincere student will find that the book is a ready material for the University exams and he/she can readily answer any questions set from the various topics presented in the book. The book has been written in such a format, that answers to various questions that may be posed are a ready reference in the sections covered.

Also, keeping in mind, the fact that, a mere knowledge of theory will not fully help the students in solving numerical problems, solved examples are given at the end of the chapters for the students to understand how to go about solving the various types of problems that may be set from the different sections. A number of unsolved numerical and sample questions are also given so that the students can practice and also refer to, respectively.

Although a humble attempt has been made to present the subject matter in a simple and concise manner by keeping in mind the average students, however, no originality is claimed, or indeed can be claimed in a work which is of a specific course of this nature, as the theories and approach to the topics cannot be presented in a completely different format, different from the others. In fact, there is simply no other way.

My thanks are also due to the publishers of this volume who have taken interest and for the meticulous care taken for the publication of this volume in this wonderful form. I also thank the Principal, Shillong College, who has greatly encouraged and expressed his confidence in me to bring out this volume. I am also particularly thankful to Dr. S. Khongwir who has greatly helped me and without whose help this volume might not have been possible. Also, I thank my friends, my colleagues in the department for their relentless support and encouragement. Last but not the least I thank my students, whose sincerity has motivated me in writing this book.

I dedicate this book to my parents and I thank God Almighty for his guidance throughout and ever.

Any suggestions toward the improvement of the book will be gratefully accepted.

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STUDENTS' PUBLICATIONS
IEWDUH (Bara Bazar), Shillong - 793002
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Ph: 0364-2243 384 • Mob: +91 94361 03648